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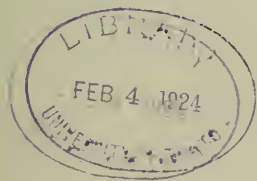
OF THE

COLLEGE OF ENGINEERING,

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Nouvelle étude relative aux fondations sur terrain élastique.

Par

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(Received July 5th 1919)

A la suite de la publication, dans le Vol. I. No. 4, 1918 de ces mémoires, d'une étude sur les fondations sur terrain élastique, nous nous sommes occupés de faire de nouvelles investigations sur le même sujet, les étendant même sur certain cas déjà traités.

La présente a pour but de les exposer et ainsi de préciser d'avantage certains points traités jusqu'ici à un point de vue plus général. Nous les donnerons dans l'ordre où il nous a été donné de les faire. Par suite l'ensemble de ce travail est imparfait quant à la coordination des différentes parties. Nous espérons cependant qu'il donnera au lecteur une idée plus exacte et plus détaillée du problème, concernant les constructions sur terrain élastique.

I. Poutre supportant une charge distincte P en son milieu.

1. Soit AB une poutre reposant sur le terrain élastique.

En posant :

$$L = \left[\frac{4EJ}{lC} \right]^{\frac{1}{4}}$$

$$\frac{l}{L} = \lambda$$

$$\frac{x}{L} = \xi$$

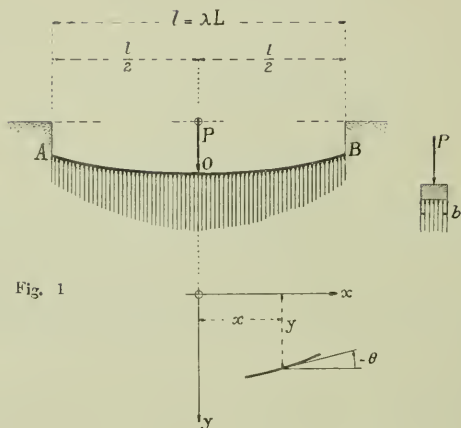


Fig. 1

nous avons pour l'équation de la courbe élastique :

$$y = -\frac{1}{2}(A_1 e^{\xi} + A_2 e^{-\xi}) \cos \xi + (B_1 e^{\xi} + B_2 e^{-\xi}) \sin \xi]$$

Les quatre constantes dans l'équation ont pour valeur :

$$(1) \begin{cases} A_1 = \frac{P}{2bCL} \frac{2 + e^{-\lambda} + \cos \lambda - \sin \lambda}{\sin \lambda + \sin \lambda} & B_1 = \frac{P}{2bCL} \frac{-e^{-\lambda} + \cos \lambda + \sin \lambda}{\sin \lambda + \sin \lambda} \\ A_2 = \frac{P}{2bCL} \frac{2 + e^{\lambda} + \cos \lambda + \sin \lambda}{\sin \lambda + \sin \lambda} & B_2 = \frac{P}{2bCL} \frac{e^{\lambda} - \cos \lambda + \sin \lambda}{\sin \lambda + \sin \lambda} \end{cases}$$

Si l'on pose :

$$(2) \begin{cases} a = \frac{2 + \cos \lambda - \sin \lambda + e^{-\lambda}}{\sin \lambda + \sin \lambda} \\ b = \frac{\cos \lambda + \sin \lambda - e^{-\lambda}}{\sin \lambda + \sin \lambda} \end{cases}$$

les dernières s'écrivent :

$$(1) \begin{cases} A_1 = \frac{P}{2bCL} a & B_1 = \frac{P}{2bCL} b \\ \text{bis} \quad A_2 = \frac{P}{2bCL} (2+a) & B_2 = \frac{P}{2bCL} (2-b) \end{cases}$$

Et en posant :

$$(3) \begin{cases} \eta = e^{-\xi} (\cos \tilde{\xi} + \sin \tilde{\xi}) & \mu = e^{-\xi} (\cos \tilde{\xi} - \sin \tilde{\xi}) \\ \eta' = -e^{-\xi} \sin \tilde{\xi} & \mu' = -e^{-\xi} \cos \tilde{\xi} \end{cases}$$

nous obtenons :

$$(4) \begin{cases} y = \frac{P}{2bCL} [\eta + a \cos \tilde{\xi} \cos \tilde{\xi} + b \sin \tilde{\xi} \sin \tilde{\xi}] \\ \quad = \frac{P}{2bCL (\sin \lambda + \sin \tilde{\lambda})} [\sin \tilde{\xi} \sin (\lambda - \tilde{\xi}) - \sin \tilde{\xi} \sin (\lambda - \tilde{\xi}) \\ \quad \quad + 2 \left\{ \cos \tilde{\xi} \cos \frac{\lambda}{4} \cos \left(\frac{\lambda}{2} - \tilde{\xi} \right) + \cos \tilde{\xi} \cos \frac{\lambda}{2} \cos \left(\frac{\lambda}{2} - \tilde{\xi} \right) \right\}] \\ M = \frac{PL}{4} [\mu - b \cos \tilde{\xi} \cos \tilde{\xi} + a \sin \tilde{\xi} \sin \tilde{\xi}] \\ Q = \frac{P}{4} [2\mu' + (a-b) \sin \tilde{\xi} \cos \tilde{\xi} + (a+b) \cos \tilde{\xi} \sin \tilde{\xi}] \\ tg \theta = \frac{P}{2bCL^2} [2\eta' + (a+b) \sin \tilde{\xi} \cos \tilde{\xi} - (a-b) \cos \tilde{\xi} \sin \tilde{\xi}] \end{cases}$$

On pose $\tilde{\xi}=0$ dans ces équations et l'on obtient au centre :

$$(5) \begin{cases} y_0 = \frac{P}{2bCL} (1+a) \\ M_0 = \frac{PL}{4} (1-b) \\ Q_0 = -\frac{P}{2} \end{cases}$$

Et à l'extrémité B où $\tilde{\xi} = \frac{\lambda}{2}$ on obtient :

$$(6) \begin{cases} y_B = \frac{2P}{bCL} c \\ M_B = 0 \\ Q_B = 0 \end{cases}$$

si l'on fait :

$$(7) \quad c = \frac{\cos \frac{\lambda}{2} \cos \frac{\lambda}{2}}{\sin \lambda + \sin \lambda}$$

2. Les trois quantités a , b , c figurant dans les équations sont des fonctions de λ seul.

Quand $\lambda=0$, a , c deviennent ∞ , tandis que b prend la forme indéterminée $\frac{0}{0}$. Sa vraie valeur s'évalue :

$$\begin{aligned} \lim_{\lambda \rightarrow 0} b &= \lim_{\lambda \rightarrow 0} \frac{\cos \lambda + \sin \lambda - e^{-\lambda}}{\sin \lambda + \sin \lambda} \\ &= \lim_{\lambda \rightarrow 0} \frac{-\sin \lambda + \cos \lambda + e^{-\lambda}}{\cos \lambda + \cos \lambda} \\ &= 1 \end{aligned}$$

Quand λ augmente indéfiniment, les deux premières tendent vers zéro, l'une gardant toujours le signe positif et l'autre prenant le signe positif ou négatif tour à tour.

Quant à c , elle prend la forme indéterminée $\frac{\infty}{\infty}$. On trouve sa vraie valeur :

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} c &= \lim_{\lambda \rightarrow \infty} \frac{\cos \frac{\lambda}{2} \cos \frac{\lambda}{2}}{\sin \lambda + \sin \lambda} \\ &= \lim_{\lambda \rightarrow \infty} \frac{\cos \frac{\lambda}{2} \cos \frac{\lambda}{2}}{\sin \lambda} \\ &= \lim_{\lambda \rightarrow \infty} \frac{\cos \frac{\lambda}{2}}{2 \sin \frac{\lambda}{2}} \\ &= \pm 0 \end{aligned}$$

Ainsi elle tend aussi vers zéro, prenant les signes $\pm$ alternativement.

La table suivante donne a , b , c pour diverses valeurs de λ .

λ	a	b	c	λ	a	b	c
0.0	∞	1.000	∞	2.0	0.178	0.079	0.184
0.2	9.000	0.900	2.500	2.2	0.135	0.021	0.144
0.4	4.000	0.800	1.250	2.5	0.103	-0.018	0.090
0.6	2.340	0.701	0.833	3.0	0.091	-0.079	0.016
0.8	1.513	0.603	0.621	π	0.090	-0.090	0
1.0	1.025	0.504	0.493	3.5	0.090	-0.081	-0.033
1.2	0.710	0.407	0.400	4.0	0.080	-0.054	-0.059
1.4	0.495	0.315	0.333	4.5	0.063	-0.027	-0.068
$\frac{\pi}{2}$	0.366	0.240	0.284	5.0	0.037	-0.017	-0.067
1.6	0.348	0.228	0.276				
1.8	0.246	0.149	0.228	∞	+0.0	± 0.0	± 0.0

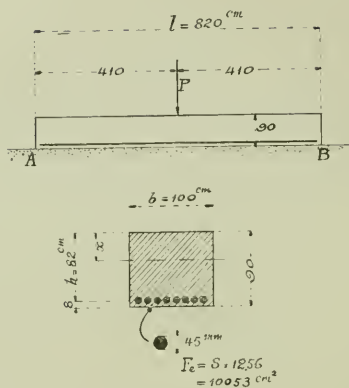
3. Prenons un exemple numérique et examinons le cas d'une poutre en béton armé, supportée par un terrain élastique et chargée d'un poids P au milieu.

Supposons $C = 15 \frac{kg}{cm^3}$.

On évalue les quantités :

$$\begin{aligned}
 x &= \frac{nF_e}{b} \left[-1 + \sqrt{1 + \frac{2bh}{nF_e}} \right] \\
 &= \frac{15 \times 100.53}{100} \left[-1 + \sqrt{1 + \frac{2 \times 100 + 82}{15 \times 100.53}} \right] \\
 &= 37 \text{ cm.}
 \end{aligned}$$

Fig. 2.



$$\begin{aligned}
 J &= \frac{bx^3}{3} + nF_i(h-x)^2 \\
 &= \frac{100 \times 37^3}{3} + 15 \times 100.53 \times 45^2 \\
 &= 4,743,000 \text{ cm}^4
 \end{aligned}$$

$$\begin{aligned}
 L &= \left[\frac{4EJ}{bC} \right]^{\frac{1}{4}} \\
 &= \left[\frac{4 \times 140,000 \times 4,743,000}{15 \times 100} \right]^{\frac{1}{4}} \\
 &= 205.134 \text{ cm}
 \end{aligned}$$

$$\lambda = \frac{l}{L} = \frac{820}{205.134} = 4.00 \text{ rad.}$$

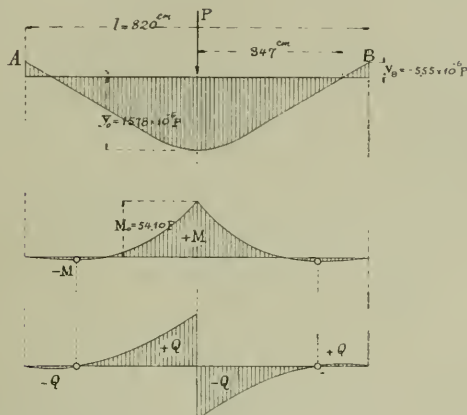
$$a = 0.080 \quad b = -0.054 \quad c = -0.059$$

Avec ces valeurs, on obtient :

$$\begin{cases}
 y = \frac{P}{0.615 \times 10^6} [e^{-\xi} (\cos \xi + \sin \xi) + 0.080 \cos \xi \cos \xi - 0.054 \sin \xi \sin \xi] \\
 M = 51.284 P [e^{-\xi} (\cos \xi - \sin \xi) + 0.054 \cos \xi \cos \xi + 0.080 \sin \xi \sin \xi]
 \end{cases}$$

$$\left\{ \begin{array}{l} Q = \frac{P}{4} [-2e^{-\xi} \cos \xi + 0.134 \sin \xi \cos \xi + 0.026 \cos \xi \sin \xi] \\ y_B = -0.377 \times 10^{-6} l^3 \end{array} \right.$$

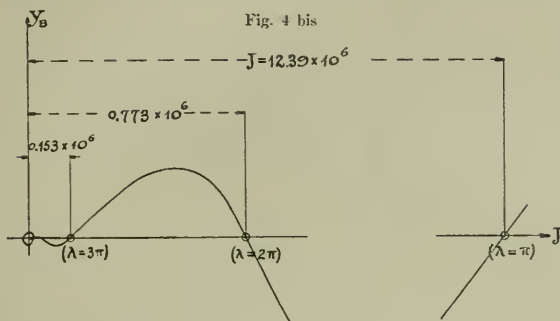
Fig. 3.



y devient négatif vers l'extrémité. Cela tient à ce que nous avons considéré la poutre comme ayant le même moment d'inertie par rapport à l'axe neutre sans avoir égard à son mode de flexion, vers le haut ou le bas et de plus que nous avons supposé que le terrain pouvait offrir aussi dans certains cas à la poutre une résistance négative. Nous reviendrons sur ce point.

Etudions le problème lorsque le moment d'inertie J varie. La quantité L s'évalue alors par l'équation

$$\begin{aligned} L &= \left[\frac{4EJ}{bC} \right]^{\frac{1}{3}} \\ &= \left[\frac{4 \times 140,000J}{100 \times 15} \right]^{\frac{1}{3}} \\ &= 4.40J^{\frac{1}{3}} \end{aligned}$$



$$\begin{cases} A_1 = \frac{P}{2bCl} a = \frac{P}{bCl} * \\ A_2 = \frac{P}{bCl} \\ B_1 = 0 \\ B_2 = 0 \end{cases}$$

Par suite on a :

$$(S) \begin{cases} y = \frac{1}{2} (A_1 + A_2) \\ = \frac{P}{bCl} \\ p = Cy = \frac{P}{bl} \end{cases}$$

Et en remarquant que :

* La quantité a est égale à ∞ lorsque $\lambda=0$. Mais ici l'expression $L(\sin \lambda + \sin \lambda)$ figurant dans le dénominateur de A_1 prend la forme indéterminée $\infty \times 0$. On évalue sa vraie valeur :

$$\begin{aligned} \lim_{L \rightarrow \infty} L(\sin \lambda + \sin \lambda) &= \lim_{L \rightarrow \infty} L \left(\sin \frac{l}{L} + \sin \frac{l}{L} \right) \\ &= \lim_{L \rightarrow \infty} l \left(\cos \frac{l}{L} + \cos \frac{l}{L} \right) \\ &= 2l. \end{aligned}$$

$$\left\{ \begin{array}{l} \lim_{L \rightarrow \infty} I_1^2 [B_1 e^{\frac{x}{l}} - B_2 e^{-\frac{x}{l}}] \cos \xi = -\frac{P}{2bC} (l - 2x) \\ \lim_{L \rightarrow \infty} I_2^2 [A_1 e^{\frac{x}{l}} - A_2 e^{-\frac{x}{l}}] \sin \xi = -\frac{P}{2bC} 2x \left(l - \frac{2x}{l} \right) \\ \lim_{L \rightarrow \infty} a (e^{\frac{x}{l}} - e^{-\frac{x}{l}}) = \frac{4x}{l} \\ \lim_{L \rightarrow \infty} a \sin \xi = \frac{2x}{l} \end{array} \right.$$

nous obtenons :

$$(9) \left\{ \begin{array}{l} M = \frac{P}{2l} \left[\frac{l}{2} - x \right]^2 \\ Q = P \left[\frac{x}{l} - \frac{1}{2} \right] \end{array} \right.$$

Les formules obtenues (9) montrent que le cas traité est précisément ce qu'on considère dans la pratique en supposant que la charge P se répartisse uniformément sur toute la longueur de la poutre.

Pour l'exemple numérique pris, les équations deviennent :

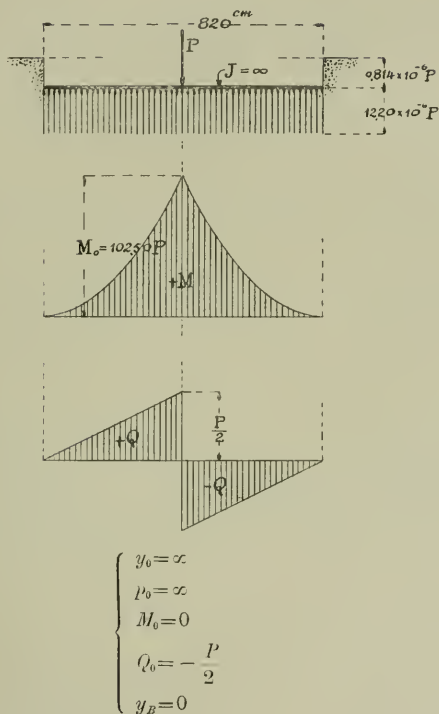
$$\left\{ \begin{array}{l} y = 0.814 \times 10^{-6} P \\ p = 12.20 \times 10^{-6} P \\ M = 6.100(410 - x)^2 \times 10^{-4} P \\ Q = 1.220(x - 410) \dots 10^{-3} P \end{array} \right.$$

On voit que le moment fléchissant M_0 est au maximum lorsque J augmente indéfiniment, tandis que le déplacement y_0 est alors au minimum (Fig. 5).

5. Soit l variable.

Supposons d'abord que la poutre soit réduite à un point, on a alors, en posant $\lambda = 0$.

Fig. 5.



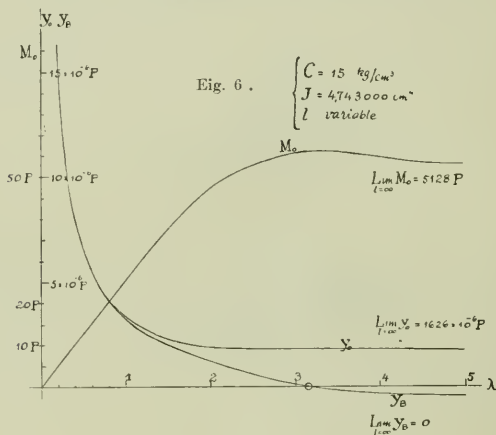
résultats qu'il était facile de prévoir.

Lorsque, au contraire, l croît indéfiniment, on a :

$$(10) \left\{ \begin{array}{l} y = \frac{P}{2bCL} \eta \\ M = \frac{PL}{4} \mu \\ Q = \frac{P}{2} \mu' \\ y_B = 0 \end{array} \right. \quad \text{et} \quad \left\{ \begin{array}{l} y_0 = \frac{P}{2bCL} \\ M_0 = \frac{PL}{4} \\ Q_0 = -\frac{P}{2} \end{array} \right.$$

Ainsi on voit que les valeurs de y_0 , M_0 dans le cas d'une poutre de longueur infinie sont les mêmes que dans le cas d'une poutre de longueur déterminée $2L$, dont le moment d'inertie serait infini.

Traçons les courbes M_0 , y_0 , y_B pour notre exemple :



6. Supposons enfin que J , l étant données, C est seul variable.

i. Si C est égale à zéro, la quantité L , qui a pour expression :

$$L = KC^{-\frac{1}{2}}$$

devient ∞ et par suite $\lambda = \frac{l}{L} = 0$.

On aperçoit donc que pour $C=0$, les quantités a, b, c qui ne sont exprimées que comme des fonctions de λ , ont les mêmes valeurs que pour $\lambda=0$, c-à-d :

$$* \quad K = \left[\frac{4EJ}{b} \right]^{\frac{1}{2}}$$

$$\begin{cases} a = \infty \\ b = 1 \\ c = \infty \end{cases}$$

On a donc pour y , l'équation

$$y = \frac{P}{bCl} = \infty$$

Comme celles de p , M , Q on peut utiliser les résultats obtenus pour $J = \infty$:

$$\begin{cases} p = \frac{P}{bl} = \text{Constante} \\ M = \frac{P}{2l} \left(\frac{l}{2} - x \right)^2 \\ Q = P \left(\frac{x}{l} - \frac{1}{2} \right) \end{cases}$$

ii. Etudions ensuite le cas où $C = \infty$. Ce cas correspond au terrain parfaitement rigide tel que le cas d'un roc solide. On voit que la quantité L , qui est toujours positive, tend vers zéro, lorsque C croît indéfiniment et que la quantité λ , qu'on peut exprimer par :

$$\lambda = \frac{lC^{\frac{1}{4}}}{K}$$

dans laquelle l n'étant pas zéro, tend vers l'infini.

Les trois quantités a , b , c ont donc les mêmes valeurs que pour $\lambda = \infty$, c-à-d :

$$\begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

Déplacements du centre et des extrémités y_0 , y_B :

L'expression de y_0 est (5)

$$y_0 = \frac{P}{2bCL} (1 + a)$$

La quantité $CL(=KC^{\frac{3}{2}})$ figurant dans le dénominateur, croît indéfiniment et par suite, on a :

$$(11)a \quad y_0 = 0$$

Le déplacement du centre y_0 , quantité toujours positive, s'approche donc de zéro, tandis que la réaction correspondante qui a pour expression.

$$(11)b \quad p_0 = \frac{P}{2bL}(1+a)$$

augmente indéfiniment.

Quant à la quantité y_B , on a :

$$(11)c \quad y_B = \frac{2P}{bCL} \cdot c \\ = 0$$

L'expressions de $p_B(=Cy_B)$, prend la forme indéterminée $\frac{0}{0}$. En négligeant $\sin \lambda$ devant $\sin \lambda$, elle s'écrit :

$$p_B = \frac{2P}{bL} \cdot \frac{\cos \frac{\lambda}{2}}{\sin \frac{\lambda}{2}}$$

Ici la quantité $L \sin \frac{\lambda}{2}$ dans le dénominateur est de nouveau de la forme indéterminée $0 \times \infty$. Sa vraie valeur se détermine :

$$\lim_{\lambda \rightarrow \infty} L \sin \frac{\lambda}{2} = K \lim_{C \rightarrow \infty} C^{-\frac{1}{2}} \cdot \sin \frac{lC^{\frac{1}{2}}}{2K} \\ = \lim_{C \rightarrow \infty} \frac{l}{2} \cos \frac{\lambda}{2} \\ = \infty$$

Par suite, on a :

$$(11)d \quad p_B = 0$$

Le déplacement y_B ainsi que la réaction correspondante p_B prennent les signes $\pm$ et tendent vers zéro.

Expression générale de p .

L'expression générale de p est (4)

$$p = Cy \\ = \frac{P}{2bL} [e^{-\xi}(\cos \xi + \sin \xi) + a \cos \xi \cos \xi + b \sin \xi \sin \xi]$$

La quantité variable $\xi \left(= \frac{x}{L} = \frac{xC^{\frac{1}{4}}}{K} \right)$, comme λ , tend généralement vers l'infini avec C ; mais x étant indépendant de C , la quantité $xC^{\frac{1}{4}}$ reste toujours zéro lorsque x est zéro, quelque valeur que C prenne. On voit donc que la quantité ξ est zéro exclusivement pour $x=0$ et $C=\infty$.

La quantité $\frac{e^{-\xi}}{L}$ prend donc généralement la forme indéterminée $\frac{0}{0}$, tandis qu'elle tend vers l'infini seulement pour $x=0$ et $C=\infty$.

Cherchons sa vraie valeur :

$$\begin{aligned} \lim_{C=\infty} \frac{e^{-\xi}}{L} &= \frac{1}{K} \lim_{C=\infty} C^{\frac{1}{4}} e^{-\frac{xC^{\frac{1}{4}}}{K}} \\ &= \lim_{C=\infty} e^{-\frac{xC^{\frac{1}{4}}}{K}} \\ &= 0 \quad (x \text{ n'étant pas zéro}) \end{aligned}$$

La quantité $a \cos \xi$ du second terme de l'expression général prend la forme indéterminée $0 \times \infty$, lorsque x est différent de zéro. On détermine sa vraie valeur :

$$\begin{aligned} \lim_{C=\infty} a \cos \xi &= \lim_{C=\infty} \frac{2 + \cos \lambda - \sin \lambda + e^{-\lambda}}{\sin \lambda + \sin \lambda} \cos \xi \\ &= \lim_{C=\infty} \frac{2 + \cos \lambda - \sin \lambda}{\sin \lambda} \cos \xi \end{aligned}$$

$$= \lim_{\lambda \rightarrow \infty} (2 + \cos \lambda - \sin \lambda) e^{-(\lambda - \xi)} *$$

La dernière tendant vers zéro, on détermine la vraie valeur de l'expression $\frac{a \cos \xi}{L}$, comme suit :

$$\lim_{\lambda \rightarrow \infty} \frac{a \cos \xi}{L} = \lim_{\lambda \rightarrow \infty} (2 + \cos \lambda) \frac{e^{-(\lambda - \xi)}}{L}.$$

L'expression $\frac{e^{-(\lambda - \xi)}}{L}$, comme on a vu plus haut, tendant vers zéro, la valeur cherchée tend aussi vers zéro.

De même le troisième terme de p , $\frac{b \sin \xi}{L}$ a pour vraie valeur zéro.

L'expression cherchée de p est donc :

$$(12) \begin{cases} p=0 \\ p=\infty, \text{ exclusivement pour } x=0. \end{cases}$$

Expressions générales de M , Q :

Nous avons (4)

$$\begin{cases} M = \frac{PL}{4} [\mu - b \cos \xi \cos \xi + a \sin \xi \sin \xi] \\ Q = \frac{P}{4} [2\mu' + (a-b) \sin \xi \cos \xi + (a+b) \cos \xi \sin \xi] \end{cases}$$

Les deux quantités μ , μ' dans les expressions, comme η , tendent généralement vers zéro, excepté pour $x=0$, où la première devient 1 et la seconde -1.

Quant aux quatre quantités $b \cos \xi$, $a \sin \xi$, $(a+b) \sin \xi$, $(a+b) \cos \xi$

$$\begin{aligned} * \quad \lim_{\lambda \rightarrow \infty} \frac{\cos \xi}{\sin \lambda} &= \lim_{\lambda \rightarrow \infty} \frac{e^{\xi}}{e^{\lambda}} \\ &= \lim_{\lambda \rightarrow \infty} e^{-(\lambda - \xi)} \\ &= \begin{cases} 0 \\ 1, \text{ exclusivement pour } \lambda = \xi. \end{cases} \end{aligned}$$

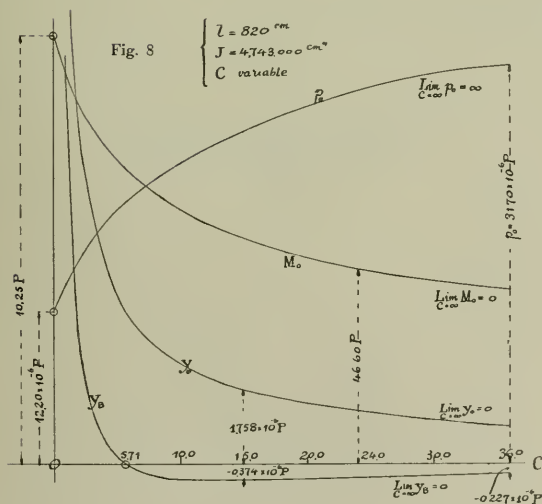
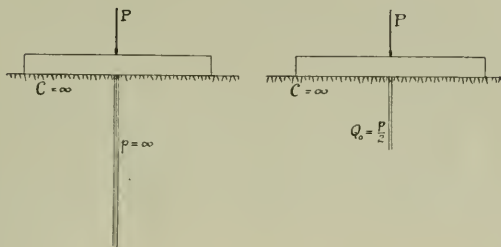
x variant de 0 à $\frac{l}{2}$, la quantité ξ est tout au plus égale à $\frac{\lambda}{2}$. Pour notre cas nous pouvons donc toujours poser

$$\lim_{\lambda \rightarrow \infty} \frac{\cos \xi}{\sin \lambda} = 0.$$

qui sont de la forme indéterminée $0 \times \infty$, elles s'annulent toutes les quatre dans notre cas. On a donc :

$$(13) \begin{cases} M=0 \\ \begin{cases} =0 \\ =-\frac{P}{2} \end{cases} \\ Q \end{cases}, \text{ exclusivement pour } x=0.$$

Fig. 7



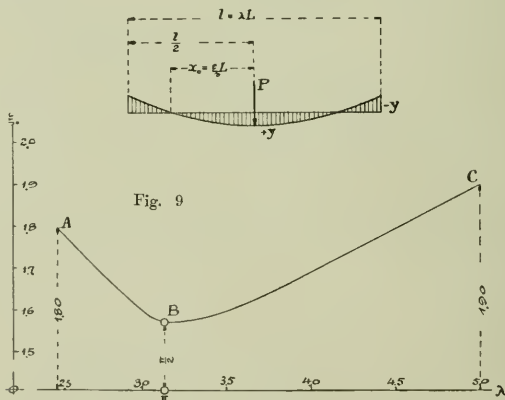
II. Étude du cas où le terrain n'offre pas de résistance négative.

1. Il est évident que, si le terrain, ainsi qu'il est en général, ne peut offrir aucune résistance négative contre la poutre, la poutre d'une longueur supérieure à πL , s'élève à ses deux extrémités et cette partie élevée ne se charge d'aucune partie de la charge extérieure.

Cherchons d'abord les points où y s'annule. On désigne par x_0 l'abscisse de ces points. Si l'on pose $\xi_0 = \frac{x_0}{L}$, toutes les valeurs de ξ_0 sont données par l'équation

$$(1) \quad \sin \xi_0 \sin(\lambda - \xi_0) - \cos \xi_0 \sin(\lambda - \xi_0) + 2 \left\{ \cos \xi_0 \cos \frac{\lambda}{2} \cos \left(\frac{\lambda}{2} - \xi_0 \right) + \cos \xi_0 \cos \frac{\lambda}{2} \cos \left(\frac{\lambda}{2} - \xi_0 \right) \right\} = 0$$

équation donnant ξ_0 en fonction de λ .



On remarque, sur la figure 9, que pour une poutre de la

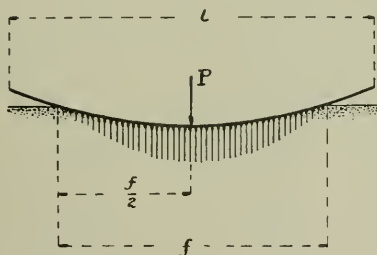
longueur πL , ξ_0 est égal à $\frac{\pi}{2}$; en d'autres termes cette poutre a son déplacement nul à ses extrémités.

Lorsque λ augmente au-delà de π , mais restant dans une certaine limite, ξ_0 devient supérieur à $\frac{\pi}{2}$; mais ξ_0 n'atteignant jamais les demi-valeurs correspondantes de λ , nous pouvons conclure que dans une poutre, portant telle valeur de λ , il y a un point où y s'annule. Pour les valeurs de λ inférieures à π , celles de ξ_0 correspondantes étant toujours supérieures à $\frac{\lambda}{2}$, la poutre n'a pas de points où le déplacement devient nul.

Il suffit donc, pour nous rendre compte des points de déplacement nul, de tracer la partie BC de la courbe.

2. La longueur f , qui transmet la charge P au terrain, se calcule donc :

Fig. 10



$$(2) \left\{ \begin{aligned} f &= \pi L \\ &= \pi \left[\frac{4EJ}{bC} \right]^{\frac{1}{4}} \\ &= \frac{4.4429}{C^{\frac{1}{4}}} \left[\frac{EJ}{b} \right]^{\frac{1}{4}} \\ &= 4.4429 \left[\frac{J}{C} \right]^{\frac{1}{4}} \left[\frac{E}{b} \right]^{\frac{1}{4}} \end{aligned} \right.$$

C	$\frac{4.4429}{C^{\frac{1}{4}}}$	C	$\frac{4.4429}{C^{\frac{1}{4}}}$
0.00	∞	8	4.642
0.03	10.675	9	2.565
0.05	9.396	10	2.498
0.10	7.901	20	2.101
0.30	6.003	30	1.898
0.50	5.284	40	1.767
0.70	4.857	50	1.675
1	4.443	70	1.536
2	3.763	100	1.405
3	3.756	300	1.068
4	3.142	500	0.940
5	2.971	700	0.864
6	2.839	1000	0.790
7	2.732	5000	0.528

$\frac{J}{C}$	$4.4429 \left[\frac{J}{C} \right]^{\frac{1}{4}}$	$\frac{J}{C}$	$4.4429 \left[\frac{J}{C} \right]^{\frac{1}{4}}$
0	0	5×10^4	66.439
10^2	14.049	10^5	79.007
5×10^2	21.009	5×10^5	118.14
10^3	24.984	10^6	140.50
3×10^3	32.881	5×10^6	210.10
5×10^3	37.360	10^7	249.84
10^4	44.429	10^8	444.29

Et en introduisant dans les équations générales les valeurs de a, b pour $\lambda=\pi$, on a :

$$(3) \left\{ \begin{array}{l} y = \frac{P}{2bCL} [\eta + 0.090 \{ \cos \xi \cos \xi - \sin \xi \sin \xi \}] \\ M = \frac{PL}{4} [\mu + 0.090 \{ \sin \xi \sin \xi + \cos \xi \cos \xi \}] \\ Q = \frac{P}{4} [2\mu' + 0.180 \sin \xi \cos \xi] \\ tg \theta = \frac{P}{2bCL} [2\eta' - 0.180 \cos \xi \sin \xi] \end{array} \right.$$

ξ	$\eta + 0.090 \{ \cos \xi \cos \xi - \sin \xi \sin \xi \}$	$\mu + 0.090 \{ \sin \xi \sin \xi + \cos \xi \cos \xi \}$	$2\mu' + 0.180 \sin \xi \cos \xi$
0.0	1.090	1.090	-2.000
0.2	1.052	0.734	-1.570
0.4	0.953	0.461	-1.166
0.6	0.810	0.272	-0.841
0.8	0.662	0.131	-0.515
1.0	0.489	0.054	-0.284
1.2	0.321	0.014	-0.120
1.4	0.149	0.001	-0.026
$\frac{\pi}{2}$	0.000	0.000	-0.000

Comme application, revenons à notre exemple traité précédemment :

La valeur f s'évalue :

$$\begin{aligned} f &= \pi L \\ &= 3.1416 \times 205.134 \\ &= 644.45 \text{ cm} \end{aligned}$$

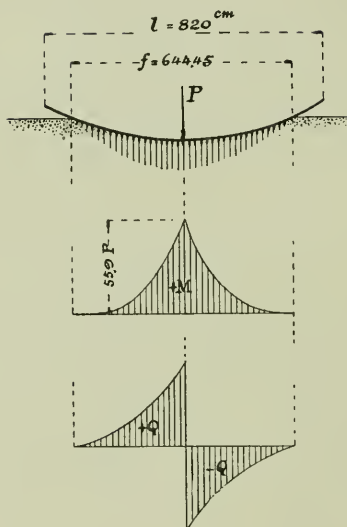
Et l'on trouve :

$$M_0 = 55.90P$$

$$y_0 = 1.774 \times 10^{-6} P$$

$$p_0 = 26.60 \times 10^{-6} P$$

Fig. 11



3. Dans ce qui précède, nous n'avons pas tenu compte du poids propre de la poutre. Il est bien naturel cependant que, lorsque la poutre commence à s'élever à ses extrémités en porte-à-faux, le poids qui n'est généralement pas négligeable, y donne une grande résistance. Cela a pour conséquence d'augmenter sensiblement la valeur f dans la pratique.

On doit modifier les équations comme suit :

L'équation de la courbe élastique pour la partie CO ayant toujours la forme :

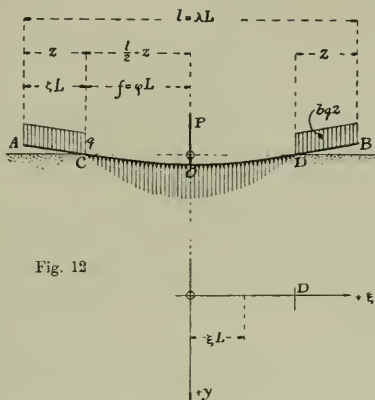


Fig. 12

$$y = \frac{1}{2} [(A_1 e^{\xi} + A_2 e^{-\xi}) \cos \xi + (B_1 e^{\xi} + B_2 e^{-\xi}) \sin \xi]$$

on a ici les cinq inconnues A_1 , A_2 , B_1 , B_2 , et z à déterminer.

Pour former les équations de condition on remarque que :

Au milieu de la courbe, $\frac{dy}{d\xi} = 0$;

au point D , $y = 0$, $M = -\frac{bqz^2}{2}$, $Q = bqz$

et
$$P + 2bzq = 2b \int_0^{\frac{1}{2}z} p dx$$

Tout cela se traduit par les formules :

$$(4) \left\{ \begin{array}{l} A_1 + B_1 - A_2 + B_2 = 0 \\ (A_1 e^{\varphi} + A_2 e^{-\varphi}) \cos \varphi + (B_1 e^{\varphi} + B_2 e^{-\varphi}) \sin \varphi = 0 \\ (B_1 e^{\varphi} - B_2 e^{-\varphi}) \cos \varphi - (A_1 e^{\varphi} - A_2 e^{-\varphi}) \sin \varphi = \frac{2qz^2}{CL^2} \\ -A_1 e^{\varphi} (\cos \varphi + \sin \varphi) + A_2 e^{-\varphi} (\cos \varphi - \sin \varphi) + B_1 e^{\varphi} (\cos \varphi - \sin \varphi) \\ \quad + B_2 e^{-\varphi} (\cos \varphi + \sin \varphi) = \frac{-4zq}{CL} \\ -A_1 + A_2 + B_1 + B_2 = \frac{2P}{bCL} \end{array} \right.$$

Malheureusement ces équations sont d'un maniement difficile

et ne se prêtent à l'application qu'au moyen d'un calcul bien laborieux.

III. Poutre dont la partie centrale a un moment d'inertie différent de celui des extrémités.

1. Pour diminuer le déplacement du centre on fait augmenter souvent le moment d'inertie de la poutre particulièrement dans la partie centrale.

Désignons par J le moment d'inertie pour l'intervalle CD ($=\frac{l}{2}$) et par J' celui des parties AC , DB .

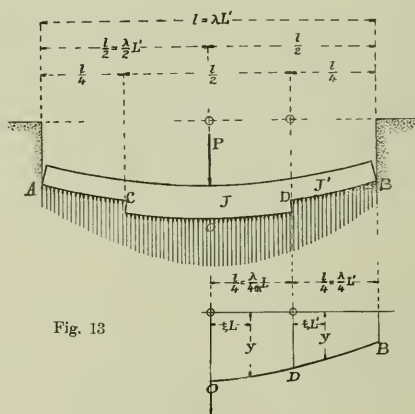


Fig. 13

Posons :

$$L = \left[\frac{4EJ}{bC} \right]^{\frac{1}{4}}$$

$$L' = \left[\frac{4EJ'}{bC} \right]^{\frac{1}{4}}$$

$$m = \frac{L}{L'} \quad \text{et} \quad \frac{\lambda}{4m} = \beta$$

Et supposons les deux équations

$$\left\{ \begin{array}{l} \text{pour la partie } OD: \quad y = \frac{1}{2} [(A_1 e^{\xi} + A_2 e^{-\xi}) \cos \frac{\xi}{2} + (B_1 e^{\xi} + B_2 e^{-\xi}) \sin \frac{\xi}{2}] \\ \text{,, ,, ,, } DB: \quad y = \frac{1}{2} [(A_1' e^{\xi} + A_2' e^{-\xi}) \cos \frac{\xi}{2} + (B_1' e^{\xi} + B_2' e^{-\xi}) \sin \frac{\xi}{2}] \end{array} \right.$$

Pour déterminer les huit inconnues nous avons :

$$(1) \left\{ \begin{array}{l} A_1 + B_1 - A_2 + B_2 = 0 \\ m[-A_1 + A_2 + B_1 + B_2] = \frac{2P}{bCL'} \\ (A_1 e^{\beta} + A_2 e^{-\beta}) \cos \beta + (B_1 e^{\beta} + B_2 e^{-\beta}) \sin \beta = A_1' + A_2' \\ A_1 e^{\beta} (\cos \beta - \sin \beta) - A_2 e^{-\beta} (\cos \beta + \sin \beta) + B_1 e^{\beta} (\cos \beta + \sin \beta) \\ \quad + B_2 e^{-\beta} (\cos \beta - \sin \beta) = m[A_1' - A_2' + B_1' + B_2'] \\ m^2[-(A_1 e^{\beta} - A_2 e^{-\beta}) \sin \beta + (B_1 e^{\beta} - B_2 e^{-\beta}) \cos \beta] = B_1' - B_2' \\ m[-A_1 e^{\beta} (\cos \beta + \sin \beta) + A_2 e^{-\beta} (\cos \beta - \sin \beta) + B_1 e^{\beta} (\cos \beta - \sin \beta) \\ \quad + B_2 e^{-\beta} (\cos \beta + \sin \beta)] = B_1' - A_1' + B_2' + A_2' \\ (B_1' e^{\frac{\lambda}{4}} - B_2' e^{-\frac{\lambda}{4}}) \cos \frac{\lambda}{4} - (A_1' e^{\frac{\lambda}{4}} - A_2' e^{-\frac{\lambda}{4}}) \sin \frac{\lambda}{4} = 0 \\ -A_1' e^{\frac{\lambda}{4}} \left(\cos \frac{\lambda}{2} + \sin \frac{\lambda}{4} \right) + A_2' e^{-\frac{\lambda}{4}} \left(\cos \frac{\lambda}{4} - \sin \frac{\lambda}{4} \right) \\ \quad + B_1' e^{\frac{\lambda}{4}} \left(\cos \frac{\lambda}{4} - \sin \frac{\lambda}{4} \right) + B_2' e^{-\frac{\lambda}{4}} \left(\cos \frac{\lambda}{4} + \sin \frac{\lambda}{4} \right) = 0 \end{array} \right.$$

dont on dresse la table à la page suivante.

2. Exemple numérique.

$$x = 30.00 \text{ cm}$$

$$J = 11,770,000 \text{ cm}^4$$

$$L = 257.465 \text{ cm}$$

$$m = \frac{L}{L'} = \frac{257.465}{205.134} = 1.255$$

$$\beta = \frac{\lambda}{4m} = \frac{4}{4 \times 1.255} = 0.80$$

$$\lambda = 4.00$$

A_1	A_2	B_1	B_2
1	-1	1	1
-1	1	1	1
$e^{\beta} \cos \beta$	$e^{-\beta} \cos \beta$	$e^{\beta} \sin \beta$	$e^{-\beta} \sin \beta$
$e^{\beta} (\cos \beta - \sin \beta)$	$-e^{\beta} (\cos \beta + \sin \beta)$	$e^{\beta} (\cos + \sin \beta)$	$e^{-\beta} (\cos \beta - \sin \beta)$
$-m^2 e^{\beta} \sin \beta$	$m^2 e^{-\beta} \sin \beta$	$m^2 e^{\beta} \cos \beta$	$-m^2 e^{-\beta} \cos \beta$
$-m e^{\beta} (\cos \beta + \sin \beta)$	$m e^{-\beta} (\cos \beta - \sin \beta)$	$m e^{\beta} (\cos \beta - \sin \beta)$	$m e^{-\beta} (\cos \beta + \sin \beta)$
0	0	0	0
0	0	0	0

A_1'	A_2'	B_1'	B_2'	
0	0	0	0	0
0	0	0	0	$\frac{2P}{bCmL'}$
-1	-1	0	0	0
$-m$	m	$-m$	$-m$	0
0	0	-1	1	0
1	-1	-1	-1	0
$-e^{\frac{\lambda}{4}} \sin \frac{\lambda}{4}$	$e^{-\frac{\lambda}{4}} \sin \frac{\lambda}{4}$	$e^{\frac{\lambda}{4}} \cos \frac{\lambda}{4}$	$-e^{-\frac{\lambda}{4}} \cos \frac{\lambda}{4}$	0
$-e^{\frac{\lambda}{4}} \left(\cos \frac{\lambda}{4} + \sin \frac{\lambda}{4} \right)$	$e^{-\frac{\lambda}{4}} \left(\cos \frac{\lambda}{4} - \sin \frac{\lambda}{4} \right)$	$e^{\frac{\lambda}{4}} \left(\cos \frac{\lambda}{4} - \sin \frac{\lambda}{4} \right)$	$e^{-\frac{\lambda}{4}} \left(\cos \frac{\lambda}{4} + \sin \frac{\lambda}{4} \right)$	0

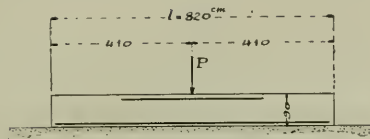
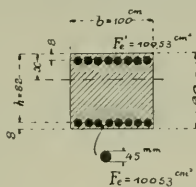


Fig. 14



On évalue ensuite :

$$\begin{aligned}
 A_1 &= 1.162 \times 10^{-7} P & A'_1 &= 0.430 \times 10^{-7} P \\
 A_2 &= 27.040 \times 10^{-7} P & A'_2 &= 16.910 \times 10^{-7} P \\
 B_1 &= -1.049 \times 10^{-7} P & B'_1 &= -2.533 \times 10^{-7} P \\
 B_2 &= 27.210 \times 10^{-7} P & B'_2 &= 2.689 \times 10^{-7} P
 \end{aligned}$$

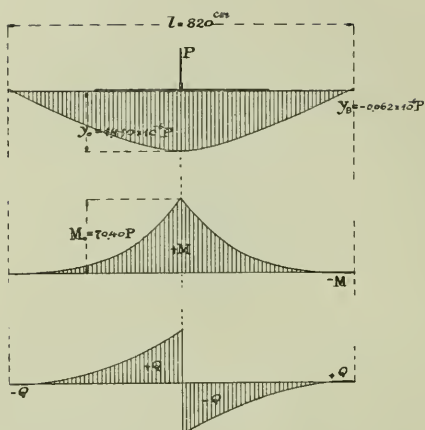
Et l'on a :

$$\begin{cases}
 y = \frac{10^{-7} P}{2} [(1.162e^{\xi} + 27.040e^{-\xi})\cos\xi - (1.049e^{\xi} - 27.210e^{-\xi})\sin\xi] \\
 M = -2.488P [(-1.049e^{\xi} - 27.210e^{-\xi})\cos\xi - (1.162e^{\xi} - 27.040e^{-\xi})\sin\xi] \\
 Q = -0.965 \times 10^{-2} P [(-2.211e^{\xi} + 54.250e^{-\xi})\cos\xi - (0.113e^{\xi} \\
 \quad - 0.170e^{-\xi})\sin\xi]
 \end{cases}$$

$$\begin{cases}
 y = \frac{10^{-7} P}{2} [(0.430e^{\xi} + 16.910e^{-\xi})\cos\xi - (2.533e^{\xi} - 2.689e^{-\xi})\sin\xi] \\
 M = 1.579P [(2.533e^{\xi} + 2.689e^{-\xi})\cos\xi + (0.430e^{\xi} - 16.910e^{-\xi})\sin\xi] \\
 Q = -0.769 \times 10^{-2} P [(-2.963e^{\xi} + 19.599e^{-\xi})\cos\xi + (2.103e^{\xi} \\
 \quad - 14.221e^{-\xi})\sin\xi]
 \end{cases}$$

On remarque sur la figure que, lorsque m croît, le déplacement au centre diminue, tandis que le moment fléchissant augmente.

Fig. 15



IV. Semelles de la fondation en maçonnerie.

1. En construisant les fondations de pile de pont ou de colonne, quand on ne peut trouver le bon sol, on donne généralement à leur base une surlargeur de chaque côté, ce qu'on appelle l'empattement, et ainsi la charge de la construction est répartie sur une surface assez large. Si les fondations sont exécutés en béton ordinaire ou en maçonnerie, l'empattement se prête à la répartition des charges, de façon à faire travailler la maçonnerie seulement en compression. Si, au contraire, on constitue l'assise en béton armé, on établit la semelle d'un hourdis de faible épaisseur, en égard à sa longueur, formant un porte-à-faux de chaque côté de la construction, laquelle doit donc supporter la réaction du terrain sur toute sa longueur et ainsi travailler en outre à la flexion.

L'intensité de la réaction n'est généralement pas la même en

tout point; elle dépend d'une part, de la dimension de l'ouvrage d'ensemble ainsi que de la manière d'armature de la semelle, et d'autre de la nature du terrain. Nous allons établir, dans ce qui suit, les équations fondamentales pour le cas général et enfin en les appliquant aux cas spéciaux, donner quelques discussions intéressantes, qui peuvent servir de guide même aux constructeurs.

2. Pour commencer, donnons une brève esquisse sur la poutre d'une section constante, supportant une charge uniformément répartie sur la partie centrale de sa longueur.

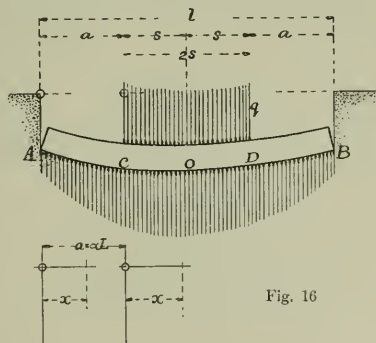


Fig. 16

$$\frac{a}{L} = \alpha$$

$$\frac{s}{L} = \sigma$$

$$\frac{2(s + a)}{L} = 2(\alpha + \sigma) = \lambda$$

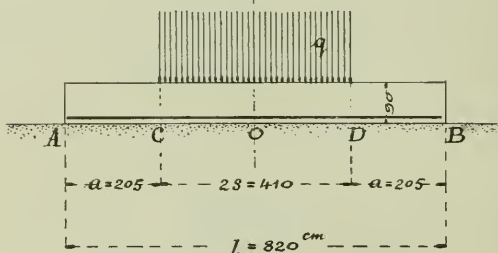
Nous supposons pour courbe élastique les deux équations.

$$\begin{cases}
 \text{pour la partie } AC, & y = \frac{1}{2} [(A_1 e^{\xi} + A_2 e^{-\xi}) \cos \tilde{\xi} + (B_1 e^{\xi} + B_2 e^{-\xi}) \sin \tilde{\xi}] \\
 \text{pour la partie } CO, & y = \frac{1}{2} [(A_1' e^{\xi} + A_2' e^{-\xi}) \cos \tilde{\xi} + (B_1' e^{\xi} + B_2' e^{-\xi}) \sin \tilde{\xi}] \\
 & + \frac{q}{C}.
 \end{cases}$$

Pour les expressions des huit quantités $A_1, A_2, B_1, B_2, A_1', A_2', B_1', B_2'$ nous renvoyons à notre ouvrage «Der Eisenbau, V. Jahrgang, No. 7».

Prenons un exemple numérique :

Fig. 18



$$\left\{ \begin{array}{l} C = 15 \frac{kg}{cm^3} \\ L = 205.134 \text{ cm} \\ E = 140,000 \frac{kg}{cm^2} \\ J = 4,743,000 \text{ cm}^4 \end{array} \right.$$

$$\left\{ \begin{array}{l} \lambda = \frac{820}{205.134} = 4 \\ a = \frac{205}{205.134} = 1 \\ \sigma = \frac{205}{205.134} = 1 \end{array} \right. \quad \left\{ \begin{array}{l} m = 7.670 \\ n = -0.309 \\ g = -8.590 \\ h = 0.607 \end{array} \right.$$

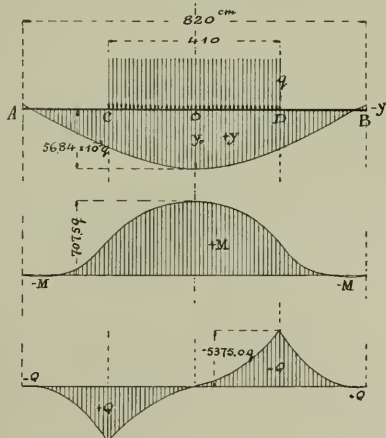
On trouve ensuite :

$$\left\{ \begin{array}{ll} A_1 = 1.655 \times 10^{-2} q & A_1' = 0.486 \times 10^{-2} q \\ A_2 = -2.690 \quad " & A_2' = -6.505 \quad " \\ B_1 = B_2 = 2.172 \quad " & B_1' = -0.735 \quad " \\ & B_2' = 1.180 \quad " \end{array} \right.$$

Et l'on obtient :

$$\begin{aligned}
 &\text{pour la partie } AC \left\{ \begin{aligned} y &= \frac{10^{-2}q}{2} [(1.655e\xi - 2.690e^{-\xi})\cos\xi + 4.344\cos\xi\sin\xi] \\ M &= -1.579 \times 10^{-5}q [4.344\sin\xi\cos\xi - (1.655e\xi + 2.690e^{-\xi})\sin\xi] \\ Q &= -769.50q [1.034\sin\xi\cos\xi - (3.827e\xi - 4.862e^{-\xi})\sin\xi] \end{aligned} \right. \\
 &\text{pour la partie } CO \left\{ \begin{aligned} y &= \frac{10^{-2}q}{2} [(0.486e\xi - 6.505e^{-\xi})\cos\xi - (0.735e\xi - 1.180e^{-\xi})\sin\xi] \\ &\quad + 6.670 \times 10^{-2}q \\ M &= -1.579 \times 10^{-5}q [(-0.735e\xi - 1.180e^{-\xi})\cos\xi \\ &\quad - (0.486e\xi + 6.505e^{-\xi})\sin\xi] \\ Q &= -769.50q [-(1.221e\xi + 5.325e^{-\xi})\cos\xi + (0.249e\xi + 7.685e^{-\xi})\sin\xi] \end{aligned} \right.
 \end{aligned}$$

17Fig. 18



Variation de y_0 :

En posant $\xi = \sigma$ dans l'expression de y , on obtient :

$$(1) \quad y_0 = \frac{q}{C} \left[1 - 2 \frac{\cos(\alpha + \sigma) \sin \alpha \cos \sigma + \cos(\alpha + \sigma) \sin \alpha \cos \sigma}{\sin 2(\alpha + \sigma) + \sin 2(\alpha + \sigma)} \right]$$

Lorsque σ est donné, y_0 est donc une fonction de α . Donnons des chiffres correspondants pour notre cas où $\sigma = 1$.

α	y_0	α	y_0
0.0	$1.000 \frac{q}{C}$	1.2	$0.856 \frac{q}{C}$
0.2	0.900 „	1.4	0.846 „
0.4	0.864 „	1.6	0.836 „
0.6	0.860 „	1.8	0.826 „
0.8	0.860 „	2.0	0.816 „
1.0	0.860 „		

3. Dans les fondations munies de semelles, les corps ont généralement un moment d'inertie très grand. On peut le considérer comme infini devant celui des semelles de chaque côté.

Considérons le cas où les deux semelles ont les mêmes longueur. Nous désignons par J leur moment d'inertie.

Le déplacement de la partie chargée étant uniforme, nous n'avons qu'à étudier l'équation de la partie DB .

$$y = \frac{1}{2} [(A_1 e^{\xi} + A_2 e^{-\xi}) \cos \xi + (B_1 e^{\xi} + B_2 e^{-\xi}) \sin \xi]$$

Nous avons maintenant les cinq inconnues A_1, A_2, B_1, B_2, y_D à déterminer. Les équations de condition se forment comme suit :

$$\text{Au point } D \ (\xi=0) \quad y=y_D, \quad \frac{dy}{d\xi}=0;$$

ce qui donne :

$$A_1 + A_2 = 2y_D$$

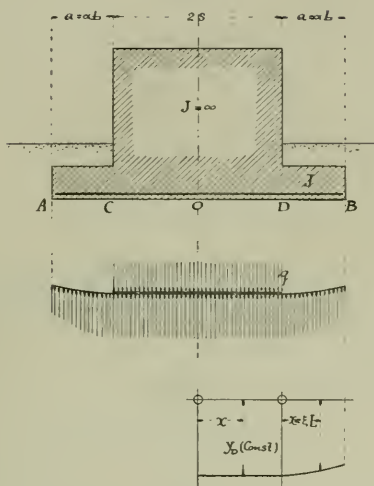
$$A_1 + B_1 - A_2 + B_2 = 0$$

$$\text{A l'extrémité } B \ (\xi=a) \quad \frac{d^2 y}{d\xi^2} = 0, \quad \frac{d^3 y}{d\xi^3} = 0;$$

ce qui donne :

$$(B_1 e^a - B_2 e^{-a}) \cos a - (A_1 e^a - A_2 e^{-a}) \sin a = 0$$

Fig. 19



$$-A_1 e^x (\cos \alpha + \sin \alpha) + A_2 e^{-x} (\cos \alpha - \sin \alpha) + B_1 e^x (\cos \alpha - \sin \alpha) + B_2 e^{-x} (\cos \alpha + \sin \alpha) = 0.$$

Et

$$2sbq = 2sbCy_D + 2b \int_0^a p \, dx,$$

ce qui donne :

$$A_1 - A_2 - B_1 - B_2 = \frac{4s}{CL} (Cy_D - q)$$

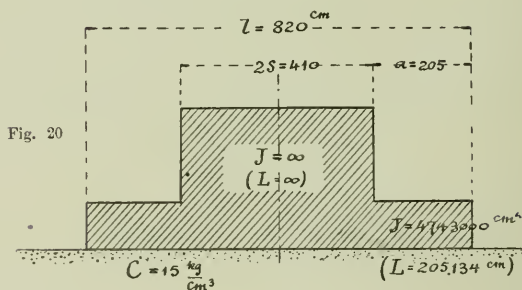
Voici la table des coefficients :

A_1	A_2	B_1	B_2	y_D	
1	1	0	0	-2	0
1	-1	1	1	0	0
$e^x \sin \alpha$	$e^{-x} \sin \alpha$	$e^x \cos \alpha$	$-e^{-x} \cos \alpha$	0	
$e^x (\cos \alpha + \sin \alpha)$	$e^{-x} (\cos \alpha - \sin \alpha)$	$e^x (\cos \alpha - \sin \alpha)$	$e^{-x} (\cos \alpha + \sin \alpha)$	0	0
1	-1	-1	-1	$-\frac{4s}{L}$	$-\frac{4sq}{CL}$

Les déterminants se développent :

$$(2) \left\{ \begin{array}{l} J = 8 \left[\sin 2\alpha + \sin 2\alpha + \frac{2s}{L} (\cos^2 \alpha + \cos^2 \alpha) \right] \\ J_{yD} = \frac{16sq}{CL} [\cos^2 \alpha + \cos^2 \alpha] \\ J_{A_1} = \frac{8sq}{CL} [-e^{-2\alpha} + 2 + \cos 2\alpha - \sin 2\alpha] \\ J_{A_2} = \frac{8sq}{CL} [e^{2\alpha} + 2 + \cos 2\alpha + \sin 2\alpha] \\ J_{B_1} = \frac{8sq}{CL} [-e^{-2\alpha} + \cos 2\alpha + \sin 2\alpha] \\ J_{B_2} = \frac{8sq}{CL} [e^{2\alpha} - \cos 2\alpha + \sin 2\alpha] \end{array} \right.$$

Prenons un exemple numérique :



$$\alpha = \frac{a}{L} = \frac{205}{205.134} = 1$$

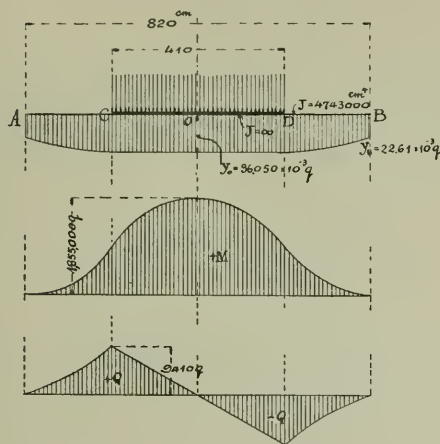
On évalue :

$$\left\{ \begin{array}{ll} y_D = 36.050 \times 10^{-3} q & B_1 = 2.416 \times 10^{-3} q \\ A_1 = 5.455 & " \quad B_2 = 58.900 \quad " \\ A_2 = 66.650 & " \end{array} \right.$$

Les équations pour la partie DB deviennent :

$$\left\{ \begin{aligned} y &= \frac{10^{-3}q}{2} [(5.455e^{\xi} + 66.660e^{-\xi})\cos\tilde{\xi} + (2.416e^{\xi} + 58.900e^{-\xi})\sin\tilde{\xi}] \\ M &= -\frac{bCL^2}{4} \times 10^{-3}q [(2.416e^{\xi} - 58.900e^{-\xi})\cos\tilde{\xi} - (5.455e^{\xi} - 66.650e^{-\xi})\sin\tilde{\xi}] \\ Q &= -\frac{bCL}{4} \times 10^{-3}q [(-3.039e^{\xi} + 125.550e^{-\xi})\cos\tilde{\xi} \\ &\quad - (7.871e^{\xi} + 7.750e^{-\xi})\sin\tilde{\xi}] \end{aligned} \right.$$

Fig. 21



Si la charge $2bsq$ est supposée concentrée à une seule charge distincte P au centre de la poutre, on a au lieu de la dernière équation de condition :

$$P = 2bsCy_D + 2b \int_0^a p \, dx$$

ce qui donne :

$$A_1 - A_2 - B_1 - B_2 = \frac{4s}{L} y_D - \frac{2P}{bCL}.$$

Pour obtenir les expressions des cinq inconnues, on n'a donc qu'à remplacer q dans les équations précédentes par $\frac{P}{2bs}$.

On aperçoit que les cinq quantités y_D, A_1, A_2, B_1, B_2 ont toutes les mêmes valeurs dans les deux cas et que par conséquent on

peut se servir pour la partie libre AC' des mêmes équations que le cas précédent. Les équations de M , Q pour la partie chargée subissent des modifications.

Le déplacement y_B à l'extrémité B a pour expression :

$$(3) \quad y_B = \begin{cases} \frac{32sq}{JCL} \cos \alpha \cos \alpha \\ \frac{16P}{JbCL} \cos \alpha \cos \alpha \end{cases}$$

Il s'annule, dans les deux cas, lorsque $\alpha = \frac{\pi}{2}$; si α dépasse cette limite, il prend le signe négative. La longueur efficiente de α est donc $\frac{\pi}{2}L$, quantité indépendante de s .

Quant au déplacement du corps y_D , il a pour expression :

$$(4) \quad y_D = \begin{cases} \frac{2sq}{CL} \left[\frac{\sin 2\alpha + \sin 2\alpha}{\cos^2 \alpha + \cos^2 \alpha} + \frac{2s}{L} \right]^{-1} \\ \frac{P}{bCL} \left[\begin{array}{c} \text{''} \end{array} \right]^{-1} \end{cases}$$

On voit que pour une valeur donnée de s , y_D varie avec la quantité $\frac{\sin 2\alpha + \sin 2\alpha}{\cos^2 \alpha + \cos^2 \alpha}$. Cette quantité toujours positive étant croissante avec α , le déplacement du corps est aussi positif et diminue lorsque la longueur α augmente.

α	$\frac{\sin 2\alpha + \sin 2\alpha}{\cos^2 \alpha + \cos^2 \alpha}$	α	$\frac{\sin 2\alpha + \sin 2\alpha}{\cos^2 \alpha + \cos^2 \alpha}$
0.0	0.00	1.8	1.840
0.2	0.400	2.0	1.852
0.4	0.748	2.2	1.872
0.6	1.170	2.4	1.903
0.8	1.483	2.6	1.929

1.0	1.700	2.8	1.950
1.2	1.802	3.0	1.967
1.4	1.830	3.5	1.992
$\frac{\pi}{2}$	1.835	4.0	2.000
1.6	1.835		

4. Supposons enfin que les deux semelles aient les différentes longueurs a_1 , a_2 . Nous supposons de plus que la charge distincte P se trouve à la distance s_1 de la surface gauche de la fondation.

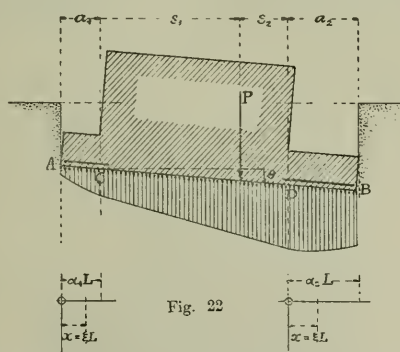


Fig. 22

La partie CD ayant la courbe élastique linéaire, nous n'avons qu'à étudier les deux équations :

$$\left\{ \begin{array}{l} \text{pour la partie } AC, \quad y = \frac{1}{2}[(A_1 e^{\xi} + A_2 e^{-\xi}) \cos \xi + (B_1 e^{\xi} + B_2 e^{-\xi}) \sin \xi] \\ \text{,, ,, ,, } DB, \quad y = \frac{1}{2}[(A_1' e^{\xi} + A_2' e^{-\xi}) \cos \xi + (B_1' e^{\xi} + B_2' e^{-\xi}) \sin \xi] \end{array} \right.$$

Nous avons ici les dix inconnues A_1 , A_2 , B_1 , B_2 , A_1' , A_2' , B_1' , B_2' , y_C , y_D à déterminer. Les équations de condition se forment par les considérations :

Aux extrémités A et B ,

$$\frac{d^2 y}{d\xi^2} = 0, \quad \frac{d^3 y}{d\xi^3} = 0;$$

au point C ,

$$y = y_C, \quad \operatorname{tg} \theta = \frac{y_D - y_C}{s_1 + s_2};$$

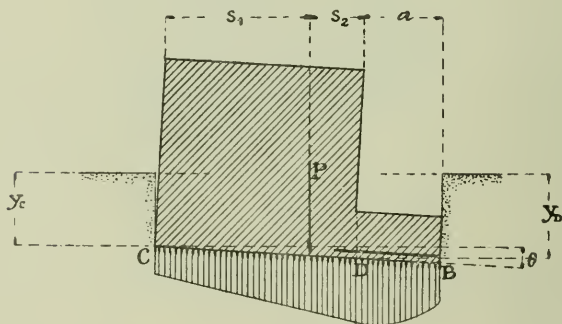
de même au point D ,

$$y = y_D, \quad \operatorname{tg} \theta = \frac{y_D - y_C}{s_1 + s_2}.$$

Les deux autres se forment en posant le moment des forces extérieures autour d'un point quelconque, par exemple le point C , et la somme des forces verticales à zéro.

En laissant de côté la solution du cas général, étudions le cas particulier $\alpha_1 = 0$.

Fig. 23



Les équations de condition deviennent :

$$\left\{ \begin{array}{l} A_1' + A_2' = 2y_D \\ A_1' + B_1' - A_2' + B_2' = 2L \frac{y_D - y_C}{s_1 + s_2} \\ B_1' e^{\alpha \cos \alpha} - B_2' e^{-\alpha \cos \alpha} - A_1' e^{\alpha \sin \alpha} + A_2' e^{-\alpha \sin \alpha} = 0 \\ -A_1' e^{\alpha} (\cos \alpha + \sin \alpha) + A_2' e^{-\alpha} (\cos \alpha - \sin \alpha) \\ \quad + B_1' e^{\alpha} (\cos \alpha - \sin \alpha) + B_2' e^{-\alpha} (\cos \alpha + \sin \alpha) = 0 \end{array} \right.$$

$$\left\{ \begin{aligned} -\frac{bCL^2}{4}[B_1' - B_2'] &= -Ps_2 + \frac{bC(s_1 + s_2)^2(y_D - 2y_c)}{6} \\ P &= \frac{bC(s_1 + s_2)(y_c + y_D)}{2} + b \int_0^a p dx^* \end{aligned} \right.$$

En prenant au lieu de y_c la différence $y_D - y_c$ comme une inconnue, nous dressons la table :

A_1'	A_2'	B_1'	B_2'	y_D	$y_D - y_c$	
1	1	0	0	-2	0	0
1	-1	1	1	0	$-\frac{2L}{s_1 + s_2}$	0
$-e^{\alpha} \sin a$	$e^{-\alpha} \sin a$	$e^{\alpha} \cos a$	$-e^{-\alpha} \cos a$	0	0	0
$-e^{-\alpha}(\cos a + \sin a)$	$e^{-\alpha}(\cos a - \sin a)$	$e^{\alpha}(\cos a - \sin a)$	$e^{-\alpha}(\cos a + \sin a)$	0	0	0
0	0	1	-1	$\frac{2(s_1 + s_2)}{L^2}$	$\frac{4(s_1 + s_2)}{3L^2}$	$\frac{4Ps_2}{bCL^2}$
-1	1	1	1	$\frac{4(s_1 + s_2)}{L}$	$-\frac{2(s_1 + s_2)}{L}$	$\frac{4P'}{bCL}$

$$* b \int_0^a p dx = -\frac{bCL}{4} [-A_1' + A_2' + B_1' + B_2']$$

On développe les déterminants :

$$(5) \begin{cases} J = -\frac{16}{3\sigma} [\sigma^4(\cos^2\alpha + \cos^2\alpha) + 2\sigma^3(\sin 2\alpha + \sin 2\alpha) + 6\sigma^2(\cos^2\alpha - \cos^2\alpha) \\ \quad + 3\sigma(\sin 2\alpha - \sin 2\alpha) + 3(\sin^2 - \sin^2\alpha)] \\ J_{y_D - y_C} = \frac{16P}{bCL} [2(\cos^2\alpha - \cos^2\alpha) + 2\sigma_2(\sin 2\alpha + \sin 2\alpha) \\ \quad - 2(\sigma_1^2 - \sigma_2^2)(\cos^2\alpha + \cos^2\alpha)] \\ J_{y_D} = \frac{16P\sigma_2}{bCL} \left[\frac{2\sigma(\sigma_2 - 2\sigma_1)}{3\sigma_2} (\cos^2\alpha + \cos^2\alpha) - \frac{1}{\sigma} \left\{ 2(\cos^2\alpha - \cos^2\alpha) \right. \right. \\ \quad \left. \left. + \frac{1}{\sigma_2} (\sin 2\alpha - \sin 2\alpha) \right\} \right] \end{cases}$$

On a ensuite :

$$y_D - y_C = \frac{J_{y_D - y_C}}{J}$$

$$y_D = \frac{J_{y_D}}{J}$$

et

$$\theta = \frac{J_{y_D - y_C}}{sJ}$$

Si l'on fait $\alpha=0$ dans ces dernières, on obtient :

$$[y_C]_{\alpha=0} = \frac{2P(2s_1 - s_2)}{bC(s_1 + s_2)^2}$$

$$[\theta]_{\alpha=0} = \frac{6P(s_1 - s_2)}{bCs^3}$$

ce qui vérifie les expressions (6) (7) du chapitre suivant.

Cherchons enfin la valeur de α , pour que l'ouvrage subisse un déplacement vertical uniforme.

Pour cela posons :

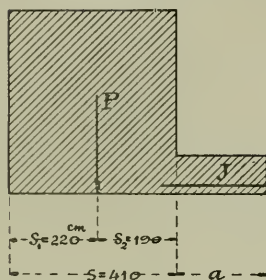
$$y_D - y_C = 0;$$

ce qui donne l'équation :

$$(6) \quad 2(\cos^2\alpha - \cos^2\alpha) + 2\sigma_2(\sin 2\alpha + \sin 2\alpha) - 2(\sigma_1^2 - \sigma_2^2)(\cos^2\alpha + \cos^2\alpha) = 0.$$

Ainsi on voit que α dépend de la quantité L . La longueur α varie donc avec la quantité L , ou avec le moment d'inertie J .

Prenons un exemple numérique :



J	L	a	$a = La$
cm^4 2808	cm 32	rad 2.160	cm 69.10
3580	34	1.082	36.80
5590	38	0.873	33.16
6860	40	0.801	32.04
16740	50	0.615	30.72
109800	80	0.399	31.92
2.68×10^5	100	0.310	31.00
2.17×10^7	300	0.100	30.00
1.67×10^8	500	0.058	28.80
2.68×10^9	1000	0.030	30.00
2.17×10^{11}	3000	0.010	30.00

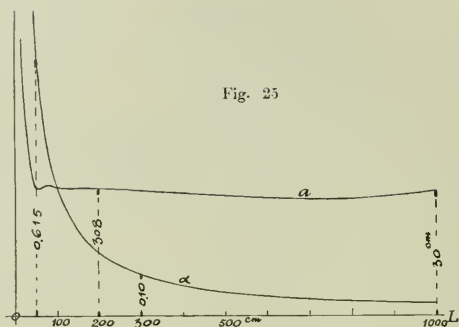
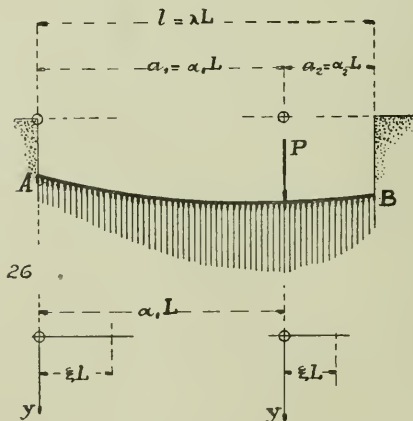


Fig. 25

On aperçoit que, lorsque J augmente indéfiniment, la quantité a s'approche de 30cm. , ce qui amène la charge P au milieu de la base.

V. Poutre supportant une charge locale en un point quelconque de sa longueur.

Fig. 26



$$\frac{a_1}{L} = a_1$$

$$\frac{a_2}{L} = a_2$$

$$\frac{l}{L} = a_1 + a_2 = \lambda$$

Supposons pour la courbe de flexion les deux équations suivantes :

$$\left\{ \begin{array}{ll} \text{pour le segment } AC, & y = \frac{1}{2} [(A_1 e^{\xi} + A_2 e^{-\xi}) \cos \xi + (B_1 e^{\xi} + B_2 e^{-\xi}) \sin \xi] \\ \text{" " " } CB, & y = \frac{1}{2} [(A_1' e^{\xi} + A_2' e^{-\xi}) \cos \xi + (B_1' e^{\xi} + B_2' e^{-\xi}) \sin \xi] \end{array} \right.$$

$$\left\{ \begin{array}{l} A_1 = \frac{-4P}{bCLJ} [(ng - mh + n - m)(e^{-\alpha_1} \sin \alpha_1 - \cos \alpha_1 \sin \alpha_1) + (m + n + g + h + 2)e^{-\alpha_1} \cos \alpha_1 - (g + h - m - n + 2) \sin \alpha_1 \cos \alpha_1 \\ \quad + (m + n) \cos \alpha_1 \cos \alpha_1] \\ \quad = \frac{-4P}{bCLJ \sin^2 \alpha_2} [2 \cos^2 \alpha_2 + \cos^2 \alpha_2)(e^{-\alpha_1} \sin \alpha_1 - \cos \alpha_1 \sin \alpha_1) + e^{-\alpha_1} \cos \alpha_2 - \cos 2\alpha_2 + e^{-2\alpha_2} \\ \quad \quad - \sin \alpha_1 \cos \alpha_1 (e^{\alpha_2} - \cos 2\alpha_2 - \sin 2\alpha_2) + (\sin 2\alpha_2 - \sin 2\alpha_2 - \sin 2\alpha_2) \cos \alpha_1 \cos \alpha_1] \\ A_2 = \frac{-4P}{bCLJ} [(ng - mh + n - m)(e^{\alpha_1} \sin \alpha_1 - \cos \alpha_1 \sin \alpha_1) - (g + h - m - n + 2) \cos \alpha_1 - (g + h + m + n + 2) \sin \alpha_1 \cos \alpha_1 \\ \quad + (m + n) \cos \alpha_1 \cos \alpha_1] \\ \quad = \frac{-4P}{bCLJ \sin^2 \alpha_2} [2(\cos^2 \alpha_2 + \cos^2 \alpha_2)(e^{\alpha_1} \sin \alpha_1 - \cos \alpha_1 \sin \alpha_1) - e^{\alpha_1} \cos \alpha_1 (-\sin 2\alpha_2 - \cos 2\alpha_2 + e^{-2\alpha_2}) \\ \quad \quad - \sin \alpha_1 \cos \alpha_1 (e^{-\alpha_2} + \sin 2\alpha_2 - \cos 2\alpha_2) + (\sin 2\alpha_2 - \sin 2\alpha_2) \cos \alpha_1 \cos \alpha_1] \end{array} \right.$$

$$\begin{aligned}
 (1) \quad & B_1 = B_2 = \frac{-4P}{bCL} [\zeta \cos a_1 \{ (g+h+2) \cos a_1 + (m+n) \sin a_1 \} + \zeta \sin a_1 \{ (m-n+mb-ng) \sin a_1 - (m+n) \cos a_1 \}] \\
 & = \frac{-4P}{bCL \sin^2 a_2} [\cos a_1 \zeta \cos (a_1 + 2a_2) - \zeta \cos a_1 \cos (a_1 + 2a_2) - 2 \cos a_2 \zeta \sin a_1 \sin \lambda - 2 \zeta \cos a_2 \sin a_1 \zeta \sin \lambda] \\
 & A'_1 = A_1 e^{\alpha_1} \cos a_1 + B_1 e^{\alpha_1} \sin a_1 - \frac{P}{bCL} \\
 & = \frac{m+g}{2} B'_1 + \frac{n+h}{2} B'_2 \\
 & A'_2 = A_2 e^{-\alpha_2} \cos a_1 + B_2 e^{-\alpha_2} \sin a_1 + \frac{P}{bCL} \\
 & = \frac{m-g}{2} B'_1 + \frac{n-h}{2} B'_2 \\
 & B'_1 = -A_1 e^{\alpha_1} \sin a_1 + B_1 e^{\alpha_1} \cos a_1 + \frac{P}{bCL} \\
 & B'_2 = -A_2 e^{\alpha_2} \sin a_1 + B_2 e^{\alpha_2} \cos a_1 + \frac{P}{bCL}
 \end{aligned}$$

dans lesquelles

$$\left\{ \begin{aligned} J &= \frac{8(\zeta \sin^2 \lambda - \sin^2 \lambda)}{\sin^2 a_2} \\ * (1)_{\text{bis}} & \\ m &= \frac{\sin 2a_1 - 1 - e^{2\alpha_2}}{2 \sin^2 a_2} \end{aligned} \right.$$

$$* \quad \left\{ \begin{aligned} m+n &= \frac{\sin 2a_2 - \zeta \sin 2a_2}{\sin^2 a_2} \\ n-m &= \frac{1 + \zeta \cos 2a_2}{\sin^2 a_2} \end{aligned} \right.$$

$$\left\{ \begin{array}{l} n = \frac{\sin 2a_2 + 1 + e^{-2x_2}}{2\sin^2 a_2} \\ g = \frac{\sin 2a_2 - 1 + e^{2x_2}}{2\sin^2 a_2} \\ n = \frac{-\sin 2a_2 - 1 + e^{-2x_2}}{2\sin^2 a_2} \end{array} \right.$$

Ces expressions introduites dans l'équation

$$tg\theta_c = \frac{1}{L} \left[\frac{dy}{d\xi} \right]_{\xi=x_1}$$

on obtient pour l'angle d'inclinaison au point de la charge :

$$(2) \quad \theta_c = tg\theta_c = \frac{P}{bCL^2(\Xi \sin^2 \lambda - \sin^2 \lambda)} [2(\cos^2 a_2 + \cos^2 a_2)(\sin^2 a_1 + \sin^2 a_1) - (\cos 2a_2 - \cos 2a_1)(\cos^2 a_1 + \cos^2 a_1)]$$

qui s'annule nécessairement pour $a_1 = a_2$.

En posant $\xi=0$ dans l'équation de y pour le segment AC , on obtient pour le déplacement à l'extrémité A :

$$(3) \quad \begin{aligned} y_A &= \frac{1}{2} [A_1 + A_2] \\ &= \frac{2P}{bCL} \cdot \frac{\cos a_1 \cos a_2 \Xi \sin \lambda - \cos a_2 \cos a_1 \sin \lambda}{\Xi \sin^2 \lambda - \sin^2 \lambda} \end{aligned}$$

En ramenant cette dernière à zéro, nous avons :

$$(4) \quad \cos a_1 \cos a_2 \Xi \sin \lambda - \cos a_2 \cos a_1 \sin \lambda = 0,$$

ce qui donne la relation entre les deux quantités a_1, a_2 pour une poutre de la longueur $l=\lambda L$, dont l'une des extrémité a son déplacement nul.

$$\left\{ \begin{array}{l} g+h = \frac{\cos 2x_2 - 1}{\sin^2 x_2} \\ g-h = \frac{\sin 2x_2 + \Xi \sin 2x^2}{\sin^2 x_2} \\ ng - mh = \frac{2\cos^2 x_1}{\sin^2 x_2} \end{array} \right.$$

Nous renvoyons pour l'étude détaillée de cette équation à notre mémoire cité.*

2. Examinons les cas où $J=\infty$.

La quantité L étant alors aussi ∞ , l'équation (3) devient†

$$y_A = \frac{P}{bCl} \left[\frac{\cos a_1 \cos a_2 \sin \lambda - \cos a_2 \cos a_1 \sin \lambda}{\sin \lambda - \sin \lambda} \right]$$

L'expression entre crochets prend de nouveau la forme indéterminée $\frac{0}{0}$. En faisant le coefficient des dérivées du dénominateur et du numérateur, on arrive à l'expression

$$y_A = \frac{-2P}{bCl^2(\cos \lambda - \cos \lambda)} \left[\begin{array}{l} a_1(\sin a_1 \cos a_2 \sin \lambda + \cos a_2 \sin a_1 \sin \lambda) \\ -a_2(\cos a_1 \sin a_2 \sin \lambda + \sin a_2 \cos a_1 \sin \lambda) \\ +l(\cos a_2 \cos a_1 \cos \lambda - \cos a_1 \cos a_2 \cos \lambda) \end{array} \right]$$

expression dont les termes se présentent tous sous la forme indéterminée $\frac{0}{0}$.

Mais en remarquant que :

$$\lim_{L \rightarrow \infty} \frac{\sin a_1 \sin \lambda}{\cos \lambda - \cos \lambda} = \frac{a_1}{l}$$

.....

$$\lim_{L \rightarrow \infty} \frac{\cos a_2 \cos a_1 \cos \lambda - \cos a_1 \cos a_2 \cos \lambda}{\cos \lambda - \cos \lambda} = -\frac{2a_2}{l}$$

on a en définitif

$$\begin{aligned} (5) \quad y_A &= \frac{-2P(a_1 - 2a_2)}{bCl^2} \\ &= \frac{-2P(l - 3a_2)^*}{bCl^2} \end{aligned}$$

* Der Eisenbau, V Jahrgang, No. 7.

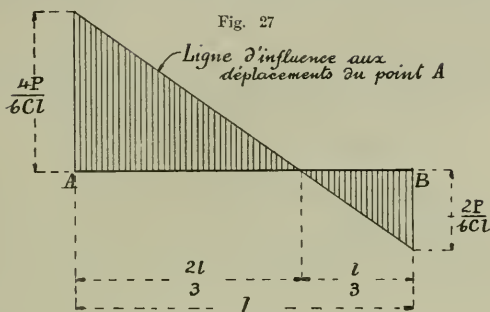
† L étant ∞ , le facteur L ($\sin \lambda + \sin \lambda$), figurant dans le dénominateur de l'expression (3) prend la forme indéterminée $\infty \times 0$.

Sa vraie valeur s'évalue :

$$\begin{aligned} \lim_{L \rightarrow \infty} L(\sin \lambda + \sin \lambda) &= \lim_{L \rightarrow \infty} l(\cos \lambda + \cos \lambda) \\ &= 2l. \end{aligned}$$

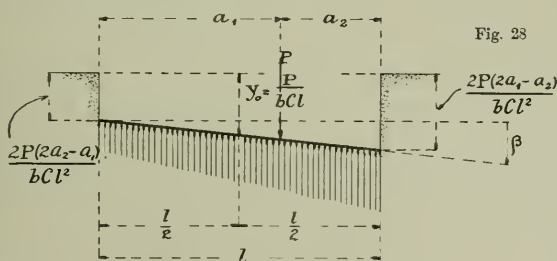
* Pour $a_1 = a_2 = \frac{l}{2}$, elle devient :

$$y_A = \frac{P}{bCl}.$$



Cette expression est précisément celle que l'on obtiendrait en traitant par la méthode ordinaire le cas d'une pièce droite, dont on néglige la déformation élastique.

Pour l'angle d'inclinaison β , on a :



$$\begin{aligned}
 (6) \quad \beta = \tan \beta &= \frac{1}{l} \left[\frac{2P(2a_1a_2)}{bCl^2} - \frac{2P(2a_2 - a_1)}{bCl^2} \right] \\
 &= \frac{6P(a_1 - a_2)}{bCl^3}
 \end{aligned}$$

qui, en remarquant que :

s'écrit :

$$\begin{aligned}
 (6)_{bis} \quad \beta &= \frac{12M_0}{bCl^3}
 \end{aligned}$$

Le déplacement vertical moyen est :

$$(7) \quad y_0 = \frac{P}{bCl}$$

VI. Influence du déplacement de la culée du à l'élasticité du terrain sur les conditions de stabilité des ponts en arc.

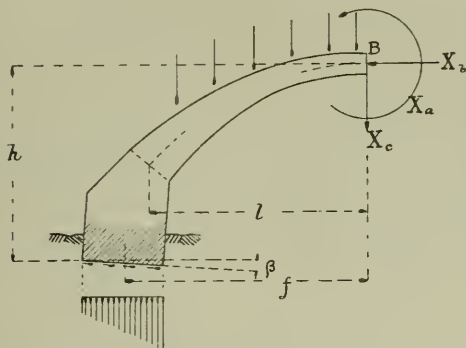
1. La théorie de la stabilité des ponts en arc est basée sur le tracé de la courbe des pressions qui suppose l'invariabilité absolue des appuis sur lesquels les ouvrages reposent. Cette hypothèse n'est pas toujours dans la pratique réalisée d'une manière parfaite. Les ouvrages s'appuient en général sur un massif de maçonnerie qui constitue la culée. La poussée de voûte transmise à ce massif se compose du poids de la culée et exerce sur le terrain de fondation une résultante, laquelle ne coupe généralement pas la base de la culée au centre. Si l'on applique à cet ouvrage les règles qui régissent la répartition des pressions sur un massif de maçonnerie comprimé, on reconnaît immédiatement qu'il devra pencher d'un côté, en se déplaçant dans le sens de la poussée. Ce mouvement entraîne forcément un déplacement angulaire de la section de naissance ainsi qu'un changement de l'ouverture de la voûte. Ainsi on ne doit pas perdre de vue que le déplacement de la culée, comme les autres sollicitations extérieures, peut avoir de grandes influences sur les fatigues dans les constructions. Les constructions peuvent se fissurer et s'ouvrir, par le seul fait du déplacement élastique de la culée, alors même que celles-ci ne supportent en aucun point un travail excessif et ne présentent aucune trace de fatigue.

Dans ce chapitre nous allons examiner pour certains cas particuliers l'influence que peut avoir le déplacement d'une culée sur les conditions de stabilité d'un pont en arc.

1. Nous coupons l'arc en deux par une section verticale

B à la clef (Fig 29). Nous avons au centre de gravité de cette section les composantes X_a et X_c de la poussée à la clef et le moment X_b de cette poussée.

Fig. 29.



Le déplacement angulaire de la semelle β est alors représenté par l'expression.

$$(1) \quad \beta = \beta_0 - X_a \beta_a - X_b \beta_b - X_c \beta_c$$

dans laquelle

β_0 = le déplacement quand les tronçons sont soumis
à la seule condition virtuelle $X = 0$, ou pour mieux
dire : $X_a = 0$, $X_b = 0$, $X_c = 0$,

$\beta_a = \dots$ à la seule condition $X_a = -1$,

$\beta_b = \dots$ „ „ „ „ „ $X_b = -1$,

$\beta_c = \dots$ „ „ „ „ „ $X_c = -1$,

Pour déterminer les trois quantités statiquement indéterminées X_a , X_b , X_c nous avons pour chaque tronçon les trois équations suivantes :

$$(2)^1 \begin{cases} L_a + \partial_a = \partial_{a0} - \partial_{aa} X_a - \partial_{ba} X_b - \partial_{ca} X_c \\ L_b + \partial_b = \partial_{b0} - \partial_{ab} X_a - \partial_{bb} X_b - \partial_{cb} X_c \\ L_c + \partial_c = \partial_{c0} - \partial_{ac} X_a - \partial_{bc} X_b - \partial_{cc} X_c \end{cases}$$

3. Considérons d'abord le cas où le pont est chargé symétriquement. Nous limiterons notre étude au cas de charge vertical.²

Comme les réactions verticales produites par cette charge sont les mêmes que celle de la poutre droite correspondante à l'arc, la force X_c devient nulle, quand la charge est symétrique. On a donc, pour les deux tronçons, à envisager les deux équations.

$$\begin{cases} L_a + \partial_a = \partial_{a0} - X_a \partial_{aa} - X_b \partial_{ba} \\ L_b + \partial_b = \partial_{b0} - X_a \partial_{ab} - X_b \partial_{bb} \end{cases}$$

dans lesquelles L_a , L_b , ont pour expression :

$$(3) \begin{cases} L_a = -[h_i \beta_0 - X_a h_i \beta_a - X_b h_i \beta_b] \\ L_b = -[\beta_0 - X_a h_i \beta_b - X_b \beta_b] \end{cases}$$

Les conditions virtuelles sont représentée sur la figure 30.

1) § 8. Gl.(1), Gl.(12). Müller-Breslau, Statik der Baukonstruktionen, Band II. II-Abt. 1908.

2) S'il y avait de plus, des forces horizontales ou inclinées, les valeurs de M_0 , M_{g0} et par suite celles de β_0 , ∂_{a0} , ∂_{b0} seraient différentes, mais les deux équations fondamentales ci-dessus seraient toujours applicables.

d'où on tire :

$$(5) \quad \begin{cases} X_a = \frac{(\partial_{a0} + h_i \beta_0)(\beta_b + \partial_{bb}) - (\partial_{b0} + \beta_0)(\beta_a + \partial_{ab})}{(h_i \beta_a + \partial_{aa})(\beta_b + \partial_{bb}) - (\beta_a + \partial_{ab})^2} \\ X_b = \frac{(\partial_{b0} + \beta_0)(h_i \beta_b + \partial_{aa}) - (\partial_{a0} + h_i \beta_0)(\beta_a + \partial_{ab})}{(h_i \beta_a + \partial_{aa})(\beta_b + \partial_{bb}) - (\beta_a + \partial_{ab})^2} \end{cases}$$

Ces dernières pour être évaluées n'exigent que l'évaluation préalable des quantités suivantes :

$$(6)^* \quad \left\{ \begin{array}{l} \beta_0 = \frac{12M_0}{bCs^2} \\ \beta_a = \frac{12h}{bCs^3} \\ \beta_b = \frac{12}{bCs^3} \\ \partial_{a0} = \int_0^l \frac{M_a M_0 dx}{EJ \cos \varphi} \\ \partial_{b0} = \int_0^l \frac{M_b M_0 dx}{EJ \cos \varphi} \\ \partial_{ab} = \partial_{ba} = \int_0^l \frac{M_a M_b dx}{EJ \cos \varphi} \\ \partial_{aa} = \int_0^l \frac{M_a^2 dx}{EJ \cos \varphi} \\ \partial_{bb} = \int_0^l \frac{M_b^2 dx}{EJ \cos \varphi} \end{array} \right.$$

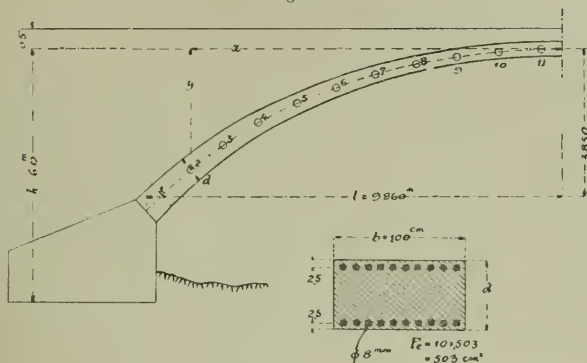
Les trois quantités β_0 , ∂_{a0} , ∂_{b0} qui ont le facteur avec l'indice 0 dans leurs expressions, varient avec la position des charges

* Voir Eq.(6) du chapitre précédent.

extérieures actuelles $P_1, P_2, \dots$, tandis que les autres en sont indépendantes et il suffit donc de les calculer une fois pour toutes.

Pour donner une idée des chiffres, prenons un pont en arc dont les dimensions principales sont :

Fig. 31.



Points*	x	y	d
1	9.545	3.550	0.700
2	8.825	2.865	0.640
3	8.045	2.270	0.590
4	7.215	1.750	0.530
5	6.305	1.300	0.490
6	5.375	0.940	0.440
7	4.425	0.625	0.400
8	3.465	0.375	0.380
9	2.485	0.200	0.360
10	1.495	0.075	0.355
11	0.500	0.000	0.350

L'armature se compose de 20 aciers de 8 mm dont 10 à l'intrados et 10 à l'extrados.

* Ces points sont choisis tels que la longueur d'arc ds devient 100 cm environ.

Nous ne prenons ici comme charges extérieures que le poids du pont lui-même.

On calcule le poids du pont en supposant qu'il se compose de béton de densité moyenne $2400 \frac{kg}{m^3}$.

On connaîtra ainsi les poids $P_1, P_2, \dots$ agissant sur les points que nous avons numérotés 1, 2, ...

Points	J	P	M_0	M_a	M_b
	cm^4	kg	$cmkg$	$cmkg$	$cmkg$
1	295×10^4	7920	13,780,000	355.00	1
2	226 "	7440	10,890,000	286.50	"
3	178 "	6338	8,425,000	227.00	"
4	129 "	5612	6,285,000	175.00	"
5	102 "	4635	4,435,000	130.00	"
6	74 "	3840	2,974,000	94.00	"
7	56 "	3288	1,889,000	62.50	"
8	48 "	2592	1,089,000	37.50	"
9	41 "	2160	545,000	20.00	"
10	39 "	1920	167,000	7.50	"
11	38 "	1680	0	0	"

Points	$\frac{M_a^2}{J}$	$\frac{M_b^2}{J}$	$\frac{M_a M_0}{J}$	$\frac{M_b M_0}{J}$	$\frac{M_a M_b}{J}$
	$^{-2}$	$^{-7}$	^{2}	$^{-2}$	$^{-4}$
1	4.273×10	3.389×10	16.55×10	467.00×10	1.202×10
2	3.635 "	4.425 "	13.81 "	482.40 "	1.268 "
3	2.895 "	5.618 "	10.73 "	472.60 "	1.275 "
4	2.895 "	7.752 "	8.54 "	487.00 "	1.358 "
5	2.375 "	9.804 "	5.64 "	434.00 "	1.275 "
6	1.658 "	13.510 "	3.78 "	402.00 "	1.270 "
7	1.194 "	17.857 "	2.11 "	337.00 "	1.117 "
8	0.697 "	20.893 "	0.85 "	226.60 "	0.781 "
9	0.293 "	24.390 "	0.27 "	132.90 "	0.488 "
10	0.098 "	25.641 "	0.03 "	42.70 "	0.192 "
11	0.014 "	26.316 "	0	0	0
Totaux	$^{-2}$ 17.132×10	$^{-9}$ 159.535×10	^{2} 62.31×16	$^{-2}$ 3484.20×10	$^{-4}$ 10.226×10

$$* \begin{cases} \delta_{aa} = \frac{17.132 \times 10^{-2} \times 100}{E} = 12.240 \times 10^{-5} \\ \delta_{bb} = \frac{159.535 \times 10^{-7} \times 100}{E} = 1.140 \times 10^{-8} \\ \delta_{a0} = \frac{62.31 \times 10^2 \times 100}{E} = 4.460 \\ \delta_{b0} = \frac{3484.20 \times 10^{-2} \times 100}{E} = 2.488 \times 10^{-2} \\ \delta_{ab} = \delta_{ba} = \frac{10.226 \times 10^{-4} \times 100}{E} = 73.00 \times 10^{-8} \end{cases}$$

Ces cinq quantités sont indépendantes de C .

La quantité M_{s0} se détermine graphiquement :

$$M_{s0} = 22.80 \times 10^6 \text{ cmkg.}$$

Enfin les trois quantités β_0 , β_a , β_b s'évaluent comme suit :

$$\begin{cases} \beta_0 = \frac{12 M_{s0}}{bCs^3} \\ \quad = \frac{12 \times 22.800 \times 10^6}{100C \times 350^3} \\ \quad = \frac{0.0638}{C} \\ \beta_a = \frac{1680 \times 10^{-6}}{C} \\ \beta_b = \frac{2800 \times 10^{-9}}{C} \end{cases}$$

C	β_0	β_a	β_b
∞	0	0	0
1000	6.380×10^{-5}	1.680×10^{-9}	2.800×10^{-12}
100	63.80 „	16.80 „	28.00 „
50	127.60 „	33.600 „	56.00 „
10	638.00 „	168.000 „	280.00 „

$$* E = 140,000 \frac{\text{kg}}{\text{cm}^2}$$

Au moyen de ces valeurs on obtient, e étant l'excentricité de la poussée X_a à la clef :

C	X_a	X_b	e
∞	$3.790 \times 10^4 \text{ kg}$	$-2.372 \times 10^5 \text{ cmkg}$	-6.26 cm
1000	3.790 „	-2.404 „	-6.35
100	3.837 „	+2.488 „	+6.50
50	3.900 „	+2.500 „	+6.41
10	4.110 „	+2.588 „	+6.30

Après avoir ainsi déterminé la poussée horizontale X_a et son point de passage, on est à même à tracer la courbe des pressions. Et en servant de la formule habituelle, on peut se rendre compte qu'aucun point de cette courbe ne se trouve en dehors du noyau du joint auquel il appartient.

Mais la présente n'ayant pour but que montrer la variation de la position de cette courbe, nous n'y avons pas mis de l'importance.

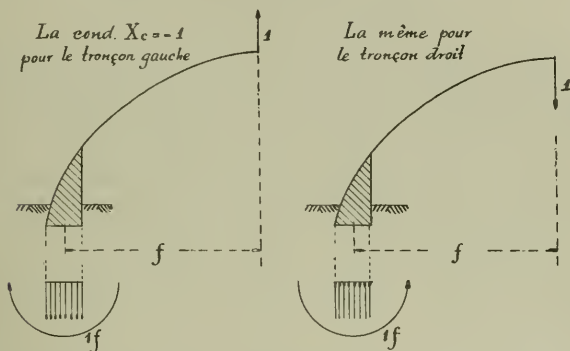
4. Nous supposons maintenant une charge uniforme $1200 \frac{\text{kg}}{\text{m}^2}$ comptée suivant l'horizontale et s'étendant sur la moitié droit de la portée.

On a ici pour chaque tronçon les trois équations à envisager, dans lesquelles.

$$(7) \begin{cases} L_a = -[h\beta_0 - X_a l_i \beta_a - X_c l_i \beta_c] \\ L_b = -[\beta_0 - X_a \beta_a - X_b \beta_b - X_c \beta_c] \\ L_c = \pm [f\beta_0 - X_a f\beta_a - X_b f\beta_b - X_c f\beta_c] \end{cases} \begin{pmatrix} + \text{ pour le tronçon gauche,} \\ - \text{ „ „ „ „ „ droit} \end{pmatrix}$$

Ici il faut distinguer des conditions virtuelles pour les deux tronçons. Le dispositif des deux conditions virtuelles $X_a = -1$, $X_b = -1$, reste le même; mais la condition virtuelle $X_c = -1$ est toute opposée en direction, ainsi :

Fig. 32.



Quant à la condition virtuelle $X=0$, le tronçon droit est chargé de la charge uniforme considérée en excès de celle du tronçon gauche.

Il est à remarquer donc que les quantités δ_{aa} , δ_{ab} , β_a qui ne dépendent que des conditions $X_a = -1$, $X_b = -1$ restent les mêmes dans les deux tronçons, tandis que celles qui dépendent de la condition $X_c = -1$, ont, à l'exception de δ_{cc} , leurs signes opposés, ainsi :

$$* \begin{cases} \delta_{ca}' = -\delta_{ca}'' \\ \delta_{cb}' = -\delta_{cb}'' \\ \beta_c' = -\beta_c'' \end{cases}$$

Ainsi on a :

$$\Sigma[\delta_{ca}] = 0, \quad \Sigma[\delta_{cb}] = 0.$$

* Nous désignons par (') les quantités appartenant au tronçon gauche et par (") celles appartenant au tronçon droit, quand elles sont de différentes valeurs dans les deux tronçons.

Et comme de plus $\Sigma[\partial_a]=0$, $\Sigma[\partial_b]=0$, $\Sigma[\partial_c]=0$, les trois équations s'écrivent :

$$(8) \quad \begin{cases} \Sigma[L_a] = \Sigma[\partial_{aa}] - 2X_a\partial_{aa} - 2X_b\partial_{ba} \\ \Sigma[L_b] = \Sigma[\partial_{bb}] - 2X_a\partial_{ab} - 2X_b\partial_{bb} \\ \Sigma[L_c] = \Sigma[\partial_{cc}] - 2X_c\partial_{cc} \end{cases}$$

En remplaçant les valeurs de L_a , L_b , L_c pour les deux tronçons et en remarquant que $\Sigma[\beta_c]=0$, $h\beta_b=\beta_a$, on obtient :

$$(8)_{\text{bis}} \quad \begin{cases} 2[h\beta_a + \partial_{aa}]X_a + 2[\beta_a + \partial_{ba}]X_b = \Sigma[\partial_{aa}] + h\Sigma[\beta_0] \\ 2[\beta_a + \partial_{ab}]X_a + 2[\beta_a + \partial_{bb}]X_b = \Sigma[\partial_{bb}] + \Sigma[\beta_0] \\ 2[f\beta_c'' + \partial_{cc}]X_c = \Sigma[\partial_{cc}] + f(\beta_0'' - \beta_0') \end{cases}$$

On en tire :

$$(9) \quad \begin{cases} X_a = \frac{(\beta_b + \partial_{bb})[\Sigma(\partial_{aa}) + h\Sigma(\beta_0)] - (\beta_a + \partial_{ab})[\Sigma(\partial_{bb}) + \Sigma(\beta_0)]}{2[h\beta_a + \partial_{aa})(\beta_b + \partial_{bb}) - (\beta_a + \partial_{ab})^2]} \\ X_b = \frac{(h\beta_a + \partial_{aa})[\Sigma(\partial_{bb}) + \Sigma(\beta_0)] - (\beta_a + \partial_{ab})[\Sigma(\partial_{aa}) + h\Sigma(\beta_0)]}{2[h\beta_a + \partial_{aa})(\beta_b + \partial_{bb}) - (\beta_a + \partial_{ab})^2]} \\ X_c = \frac{\Sigma[\partial_{cc}] + f(\beta_0'' - \beta_0')}{2[f\beta_c'' + \partial_{cc}]} \end{cases}$$

dans lesquelles

$$(10) \quad \begin{cases} \beta_c' = -\frac{12f}{bCs^3} \\ \beta_c'' = \frac{12f}{bCs^3} \end{cases}$$

et les expressions pour ∂_{c0} , ∂_{ac} , ∂_{bc} , ∂_{cc} se forment facilement

d'une manière tout analogue aux précédentes.

Revenons à notre exemple numérique :

D'abord, nous évaluons les quantités relatives au trouçon gauche. Le trouçon gauche n'ayant pour force extérieure que la charge permanente, nous n'avons qu'à nous occuper à δ_{eo} , δ_{eo} , δ_{ao} , δ_{bo} et β_c .

Points	P	M_c	$\frac{M_c M_0}{J}$	$\frac{M_c^2}{J}$	$\frac{M_a M_c}{J}$	$\frac{M_b M_c}{J}$
1	7920	<i>cm.kg</i> -954.50	-4460.00	0.309	-1149×10^{-4}	-3.236×10^{-7}
2	7440	-882.50	-4250.00	0.344	-1120 "	-3.902 "
3	6338	-804.50	-3806.00	0.363	-1025 "	-4.518 "
4	5612	-721.50	-3515.00	0.404	-179 "	-5.600 "
5	4635	-630.50	-2741.00	0.390	-804 "	-6.185 "
6	3840	-537.50	-2158.00	0.390	-684 "	-7.260 "
7	3288	-442.50	-1491.00	0.350	-494 "	-7.900 "
8	2592	-346.50	-786.00	0.250	-271 "	-7.223 "
9	2160	-248.50	-330.00	0.151	-121 "	-6.055 "
10	1920	-149.50	-64.00	0.057	-29 "	-3.835 "
11	1680	-50.00	0	0.007	0	1-.315 "
Totaux			-23601.00	3.015	-6676×10^{-4}	-57.029×10^{-7}

$$\delta_{eo} = \frac{-23601 \times 100}{E} = -16,850$$

$$\delta_{ee} = \frac{3.015 \times 100}{E} = 2.155 \times 10^{-3}$$

$$\delta_{eo} = \frac{-6676 \times 10^{-4} \times 100}{E} = -4.768 \times 10^{-4}$$

$$\delta_{eo} = \frac{-57029 \times 10^{-7} \times 100}{E} = -4.070 \times 10^{-6}$$

$$\beta_c = \frac{-12 \times 11.38}{100C \times 350^3} = \frac{-3.184 \times 10^{-6}}{C}$$

C	β_c
∞	0
1000	-3.184×10^{-9}
100	-31.84 „
50	-63.68 „
10	-318.40 „

Pour le tronçon droit, nous n'avons qu'à déterminer δ_{a0} , δ_{b0} , δ_{c0} , β_0 et β_c . Cela s'effectue comme suit :

Points	P	M_0	M_c	$\frac{M_a M_0}{J}$	$\frac{M_b M_0}{J}$	$\frac{M_c M_0}{J}$
	<i>kg</i>	<i>cmkg</i>	<i>cmkg</i>	²	⁻²	
1	8790	19,050,000	954.50	22.91×10	646.00×10	6170
2	8315	15,400,000	882.50	19.52 „	681.00 „	6020
3	7298	12,100,000	804.50	15.44 „	680.00 „	5472
4	6642	9,400,000	721.50	12.75 „	729.00 „	5260
5	5735	6,650,000	630.50	8.46 „	651.50 „	4112
6	4970	4,500,000	537.50	5.71 „	608.00 „	3268
7	4428	2,900,000	442.50	3.24 „	518.00 „	2290
8	3772	1,600,000	346.50	1.25 „	333.50 „	1155
9	3360	850,000	248.50	0.42 „	207.20 „	515
10	3120	288,000	149.50	0.06 „	73.80 „	111
11	2880	0	50.00	0	0	0
Totaux				² 89.76×10	⁻² 5128.00×10	34373

$$\delta_{e0} = \frac{89.76 \times 10^3 \times 100}{E} = 6.410$$

$$\delta_{b0} = \frac{5128.00 \times 10^{-2} \times 100}{E} = 3.664 \times 10^{-2}$$

$$\delta_{c0} = \frac{34373 \times 100}{E} = 24.520$$

La quantité M_{s0} s'évalue graphiquement :

$$M_{s0} = 30.235 \times 10^6 \text{ cmkg.}$$

Et l'on a :

$$\beta_0 = \frac{12 \times 30.235 \times 10^6}{100 \text{ C} \times 2350^3} = \frac{0.0846}{C}$$

$$\beta_c = \frac{3.184 \times 10^{-6}}{C}$$

Exposons les quantités en table.

Le tronçon gauche	Le tronçon droit
$\delta'_{a0} = 4.460$	$\delta_{a0}'' = 6.410$
$\delta'_{b0} = 2.488 \times 10^{-2}$	$\delta_{b0}'' = 3.664 \times 10^{-2}$
$\delta'_{c0} = -16.850$	$\delta_{c0}'' = 24.520$
$\delta'_{ac} = -4.768 \times 10^{-4}$	$\delta_{ac}'' = 4.768 \times 10^{-4}$
$\delta'_{bc} = -4.070 \times 10^{-4}$	$\delta_{bc}'' = 4.070 \times 10^{-6}$
$\beta_0 = \frac{0.0638}{C}$	$\beta_0'' = \frac{0.0846}{C}$
$\beta'_c = -\frac{3.184 \times 10^{-6}}{C}$	$\beta_c'' = \frac{3.184 \times 10^{-6}}{C}$
$\delta_{ab}, d_{aa}, \delta_{bb}, d_{cc}, \beta_a, \beta_b$ ont les mêmes valeurs dans les deux tronçons	

C	β'_0	β''_0	β'_c	β''_c
∞	0	0	0	0
1000	6.380×10^{-5}	8.460×10^{-5}	-3.184×10^{-9}	3.184×10^{-9}
100	63.80 „	84.60 „	-31.84 „	31.84 „
50	127.60 „	169.20 „	-63.68 „	63.68 „
10	638.00 „	846.00 „	-318.40 „	318.40 „

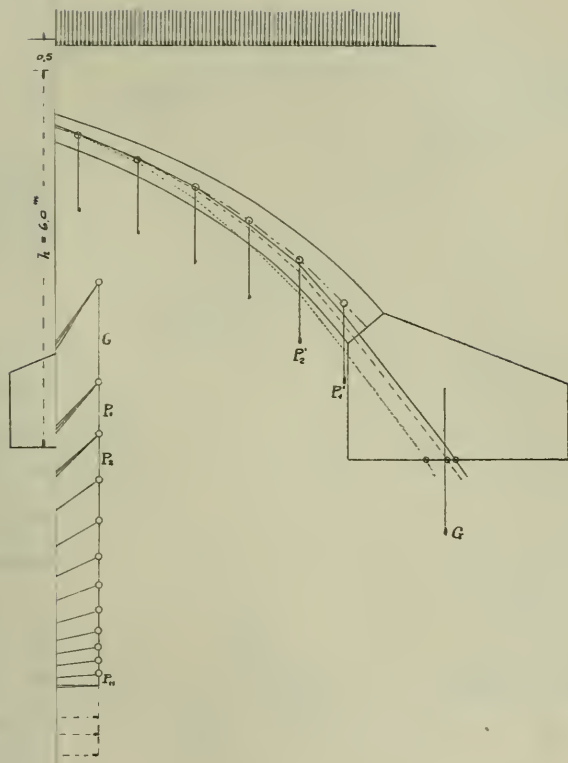
Au moyen de ces valeurs on obtient :

C	X_a	X_b	X_c	e
∞	47350^{kg}	-245000^{cmkg}	1755^{kg}	-5.17^{cm}
1000	45000	-208000	1758	-4.63
100	40900	+10430	1779	+0.25
50	37080	+224000	1800	+6.06
10	19940	+1100000	1974	+55.20

En traçant les lignes des pressions, nous remarquons que, quand la valeur C , qui est une mesure de l'élasticité du terrain, diminue, elles s'élèvent à la clef et s'abaissent aux naissances, avec diminution de la poussée à la clé.

Les valeurs X_a , X_b , X_c dépendent en général de la quantité C et de plus de la dimension principale de la construction, surtout de la hauteur de la clef h et de la distance f . On ne peut donc, rien qu'avec les formules obtenues ci-dessus, arriver à aucune conclusion absolue.

5. Dans ce qui précède, nous avons supposé que la culée ne subissait que des déplacements angulaires, c-à-d qu'elle ne



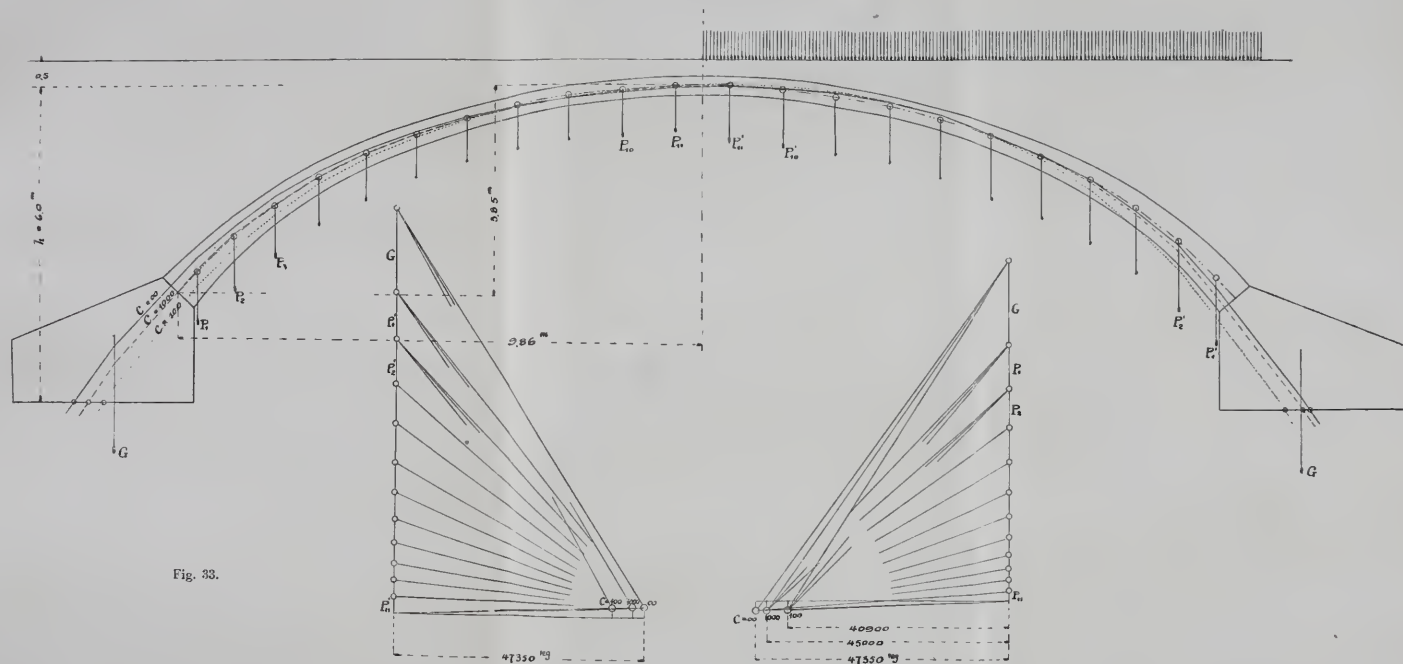
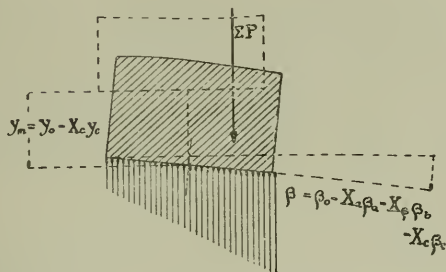


Fig. 33.

donne lieu à aucun déplacements vertical. Il est évident cependant que la construction subit, tant qu'on tient compte de l'élasticité de la fondation, les deux déplacements ensemble.

Fig. 34.



Le déplacement vertical moyen y_m a pour expression

$$(11) \quad y_m = y_0 - X_a y_a - X_b y_b - X_c y_c$$

qui, comme $y_a = 0$, $y_b = 0$, se transforme en :

$$y_m = y_0 - X_c y_c$$

Les trois quantités L_a , L_b , L_c relatives au déplacement y_m ont pour valeurs :

$$(12) \quad \begin{cases} [L_a]_y = 0 \\ [L_b]_y = 0 \\ [L_c]_y = \pm [y_0 - X_c y_c] \end{cases} \quad \begin{matrix} (+ \text{ pour le tronçon gauche} \\ - \text{ ,, ,, ,, droit} \end{matrix}$$

L'équation pour X_c , qui est de la forme

$$\Delta[L_c] = \Delta[\partial_{c0}] - 2X_c \partial_{cc}$$

se modifie seulement :

$$(13) \quad [f(\beta'_0 - \beta''_0) + 2X_c f \beta'_c'' + (y'_0 - y''_0) + 2X_c y''_c] = \Delta[\partial_{c0}] - 2X_c \partial_{cc}$$

On en tire :

$$(14) \quad X_c = \frac{\Sigma[\delta_{ee}] + f(\beta_0'' - \beta_0') + (y_0'' - y_0')}{2[f\beta_c'' + y_c'' + \delta_{ee}]}$$

Pour notre exemple,

$$* \left\{ \begin{aligned} y_c'' &= \frac{1}{bCs} = \frac{1}{100 \times 350 \text{ } C} = \frac{2.857 \times 10^{-5}}{C} \\ y_0' &= \frac{\Sigma[P]}{bCs} = \frac{63180}{100 \times 350 \text{ } C} = \frac{1.806}{C} \\ y_0'' &= \frac{75040}{100 \times 350 \text{ } C} = \frac{21.44}{C} \\ y_0'' - y_0' &= \frac{0.338}{C} \end{aligned} \right.$$

C	$y_0'' - y_0'$	y_c''	X_c
∞	0	0	1755 ^{kg}
1000	0.338×10^{-3}	2.857×10^{-8}	1758
100	3.380 „	28.57 „	1780
50	6.760 „	57.14 „	1802
10	33.80 „	285.70 „	1980

On voit ainsi que X_c , lorsque C diminue, subit une faible augmentation.

* Voir Eq.(7) du chapitre précédent.

VII. Poutre retenue en son milieu et chargée symétriquement à ses deux extrémités d'une charge uniformément répartie.

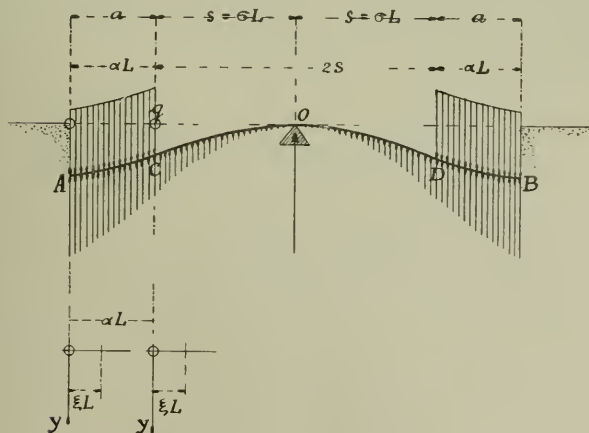
1. Soit la poutre AB ,* reposant sur un terrain élastique, chargée à ses deux extrémités d'une charge uniformément répartie.

Nous avons vu que, lorsque pour une valeur donnée de α , celle de σ † est supérieure à celle fournie par l'équation.

$$(1) \cos\sigma \cos(\alpha+\sigma) \sin\alpha + \cos\sigma \cos(\alpha+\sigma) \sin\alpha = 0,$$

il se produit dans la poutre relèvement du centre.

Fig. 35



* Voir Fig. 10, page 248, Der Eisenbau, V. Jahrgang, No. 7, 1914. ou Fig. 2, page 223, Vol. I. No. 4, 1919 de ces mémoires.

† La quantité σ est désignée par ν dans ces deux mémoires.

Supposons qu'on empêche ce relèvement par un moyen quelconque, alors nous aurons une nouvelle quantité inconnue X , qui est la réaction au point de retenue.

En supposant les deux équations

$$\left\{ \begin{array}{ll} \text{pour la partie } AC, & y = \frac{1}{2} \left[(A_1 e^{\xi} + A_2 e^{-\xi}) \cos \tilde{\xi} + (B_1 e^{\xi} + B_2 e^{-\xi}) \sin \tilde{\xi} \right] + \frac{q}{c} \\ \text{,, ,, } CO, & y = \frac{1}{2} \left[(A_1' e^{\xi} + A_2' e^{-\xi}) \cos \tilde{\xi} + (B_1' e^{\xi} + B_2' e^{-\xi}) \sin \tilde{\xi} \right] \end{array} \right.$$

on a les neuf inconnues $A_1, A_2, B_1, B_2, A_1', A_2', B_1', B_2'$ et X à déterminer.

Les équations de condition se forment presque de la même manière qu'au chapitre II. (P. 229, No. 4)

Les sept premières restent les mêmes; les deux autres se forment en considérant qu'au centre on a :

$$y=0$$

$$-\frac{bCL}{4} \cdot \frac{d^3 y}{d\tilde{\xi}^3} = -\frac{X}{2},$$

ce qui se traduit par les deux équations

$$\begin{aligned} & [A_1' e^{\tilde{\sigma}} + A_2' e^{-\tilde{\sigma}}] \cos \tilde{\sigma} + [B_1' e^{\tilde{\sigma}} + B_2' e^{-\tilde{\sigma}}] \sin \tilde{\sigma} = 0 \\ & [(B_1' - A_1') e^{\tilde{\sigma}} + (B_2' + A_2') e^{-\tilde{\sigma}}] \cos \tilde{\sigma} - [(B_1' + A_1') e^{\tilde{\sigma}} - (B_2' - A_2') e^{-\tilde{\sigma}}] \sin \tilde{\sigma} \\ & \qquad \qquad \qquad = \frac{2X}{bCL}. \end{aligned}$$

La table des coefficients est donc :

A_1	A_2	B_1	B_2	A_1'	A_2'	B_1'	B_2'	N
0	0	1	-1	0	0	0	0	0
-1	1	1	1	0	0	0	0	0
$e^{\alpha} \cos \alpha$	$e^{-\alpha} \cos \alpha$	$e^{\alpha} \sin \alpha$	$e^{-\alpha} \sin \alpha$	-1	-1	0	0	$-\frac{2q}{\epsilon'}$
$e^{\alpha} (\cos \alpha - \sin \alpha)$	$-e^{-\alpha} (\cos \alpha + \sin \alpha)$	$e^{\alpha} (\cos \alpha + \sin \alpha)$	$e^{-\alpha} (\cos \alpha - \sin \alpha)$	-1	1	-1	-1	0
$-e^{\alpha} \sin \alpha$	$e^{-\alpha} \sin \alpha$	$e^{\alpha} \cos \alpha$	$-e^{-\alpha} \cos \alpha$	0	0	-1	1	0
$e^{\alpha} (\cos \alpha + \sin \alpha)$	$e^{-\alpha} (\cos \alpha + \sin \alpha)$	$e^{\alpha} (\sin \alpha - \cos \alpha)$	$-e^{-\alpha} (\cos \alpha + \sin \alpha)$	-1	1	1	1	0
0	0	0	0	$e^{\beta} (\cos \beta - \sin \beta)$	$e^{-\beta} (\cos \beta + \sin \beta)$	$e^{\beta} (\cos \beta + \sin \beta)$	$e^{-\beta} (\cos \beta - \sin \beta)$	0
0	0	0	0	$e^{\beta} \cos \beta$	$e^{-\beta} \cos \beta$	$e^{\beta} \sin \beta$	$e^{-\beta} \sin \beta$	0
0	0	0	0	$-e^{\beta} (\cos \beta + \sin \beta)$	$e^{-\beta} (\cos \beta - \sin \beta)$	$e^{\beta} (\cos \beta - \sin \beta)$	$e^{-\beta} (\cos \beta + \sin \beta)$	$\frac{2}{\delta CL}$

Les déterminants J , J_x se développent :

$$(2) \quad \begin{cases} J = \frac{64}{bCL} [\cos^2(a+\theta) + \cos^2(a-\theta)] \\ J_x = \frac{128g}{C} [\cos\theta \cdot \sin a \cdot \cos(a+\theta) + \cos\theta \cdot \sin a \cdot \cos(a-\theta)] \end{cases}$$

2. Variation de X .

X a pour expression :

$$(3) \quad X = \frac{2bqL[\cos\theta \cdot \sin a \cdot \cos(a+\theta) + \cos\theta \cdot \sin a \cdot \cos(a-\theta)]}{\cos^2(a+\theta) + \cos^2(a-\theta)}.$$

On voit que, lorsque a est donnée, X est une fonction de θ seul. Prenons, par exemple pour le cas que nous allons donner, les chiffres correspondantes ci-dessous :

Pour $a=1.44$

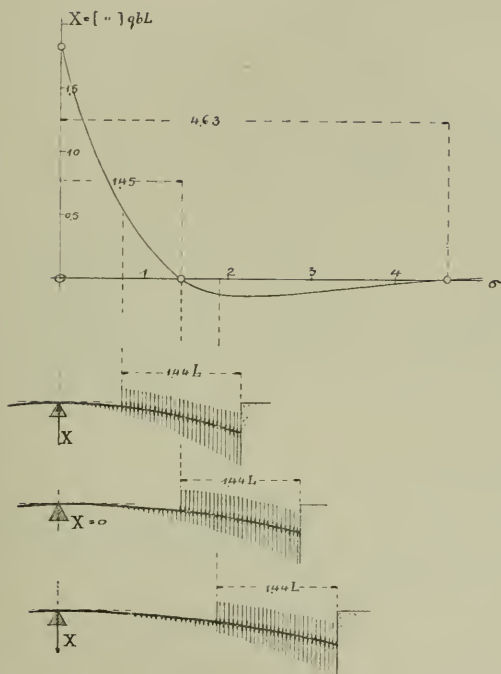
θ	X	θ	X
0	$1.834 \, bqL$	2.0	$-0.134 \, bqL$
0.5	$0.838 \quad ,,$	2.5	$-0.136 \quad ,,$
1.0	$0.296 \quad ,,$	3.0	$-0.096 \quad ,,$
1.2	$0.140 \quad ,,$	4.0	$-0.026 \quad ,,$
1.45	$0 \quad ,,$	4.5	$-0.003 \quad ,,$
1.6	$-0.056 \quad ,,$	4.63	$0 \quad ,,$
1.8	$-0.106 \quad ,,$	5.0	$+0.005 \quad ,,$

X change de signe aux points où $\theta=1.45$, $\theta=4.63$

Les points où X s'annule sont donnés en général par l'équation, en donnant au déplacement du centre la valeur zéro

$$\cos\theta \sin a \cos(a+\theta) + \cos\theta \sin a \cos(a-\theta) = 0$$

Fig. 36



équation toute semblable à celle que nous avons déjà eue*, résultat qu'il était facile à prévoir, puisque, le déplacement étant zéro, il n'exige aucun effort pour retenir le point en place.

3. Moment fléchissant au centre.

Le moment fléchissant au centre a pour équation

* Eq. (9) page 233, Memoirs of the College of Engineering, Kyushu Imp. University, Vol. I. No. 4, 1919.

$$\begin{aligned}
 M_{\xi=\bar{\sigma}} &= -\frac{bCL^2}{4} \left[\frac{d^2 y}{d\xi^2} \right]_{\xi=\bar{\sigma}} \\
 &= -\frac{bCL^2}{4} \left[(B_1' e^{\bar{\sigma}} - B_2' e^{-\bar{\sigma}}) \cos \bar{\sigma} - (A_1' e^{\bar{\sigma}} - A_2' e^{-\bar{\sigma}}) \sin \bar{\sigma} \right]
 \end{aligned}$$

En effectuant le calcul, on obtient :

$$\begin{aligned}
 (4) \quad M_{\xi=\bar{\sigma}} &= -\frac{32Lq}{AC} \left[\cos(\alpha + \bar{\sigma}) \{ \mathfrak{S} \sin \bar{\sigma} \sin \alpha - \mathfrak{C} \cos \bar{\sigma} \cos \alpha \} \right. \\
 &\quad \left. + \mathfrak{C} \cos(\alpha + \bar{\sigma}) \{ \sin \bar{\sigma} \mathfrak{S} \sin \alpha + \cos \bar{\sigma} \mathfrak{C} \cos \alpha \} \right]
 \end{aligned}$$

Pour une valeur donnée de α , $M_{\xi=\bar{\sigma}}$ est une fonction de $\bar{\sigma}$.

4. Il est à remarquer que la poutre, quand même elle est retenue au milieu, subit encore des relèvements en quelques points.

Posons

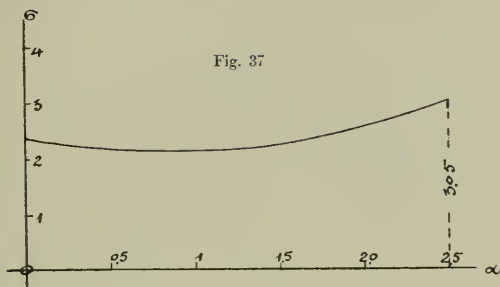
$$M_{\xi=\bar{\sigma}} = 0$$

ce qui nous donne :

$$\begin{aligned}
 (5) \quad \cos(\alpha + \bar{\sigma}) \{ \mathfrak{S} \sin \bar{\sigma} \sin \alpha - \mathfrak{C} \cos \bar{\sigma} \cos \alpha \} \\
 + \mathfrak{C} \cos(\alpha + \bar{\sigma}) \{ \sin \bar{\sigma} \mathfrak{S} \sin \alpha + \cos \bar{\sigma} \mathfrak{C} \cos \alpha \} = 0
 \end{aligned}$$

équation entre les deux quantités α et $\bar{\sigma}$.

α	$\bar{\sigma}$
0	2.360
0.1	2.320
0.5	2.175
1.0	2.170
1.4	2.230
1.6	2.270
2.5	3.050

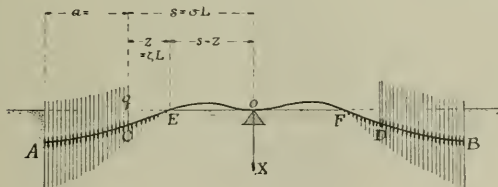


Si σ est supérieur à celle fournie par la dernière, ou $\bar{\sigma}$ est situé au-dessus de la courbe ci-dessus, $M_{\xi=\sigma}$ change de signe, ce qui veut dire qu'il se produit des relèvements au voisinage du centre.

VIII. Cas où pour une valeur donnée de α , celle de $\bar{\sigma}$ est supérieure à celle fournie par l'équation (5) du chapitre précédent.

1. Comme nous venons de remarquer, la poutre subit des relèvements au voisinage du centre.

Fig. 38

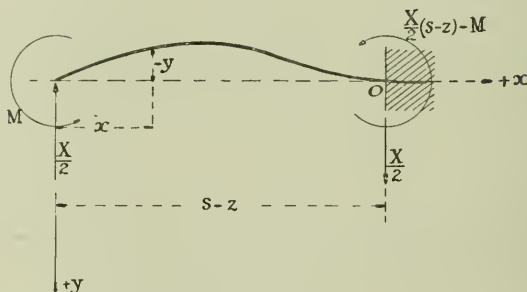


Nous supposons pour les parties AC , DE , les deux équations :

$$\begin{cases} y = \frac{1}{2} \left[(A_1 e^{\xi} + A_2 e^{-\xi}) \cos \tilde{\xi} + (B_1 e^{\xi} + B_2 e^{-\xi}) \sin \tilde{\xi} \right] + \frac{q}{c} \\ y = \frac{1}{2} \left[(A_1' e^{\xi} + A_2' e^{-\xi}) \cos \tilde{\xi} + (B_1' e^{\xi} + B_2' e^{-\xi}) \sin \tilde{\xi} \right] \end{cases}$$

Quant à celle relative à la partie élevée EO , nous la formons comme suit :

Fig. 39



Nous commençons par l'équation différentielle

$$\frac{X}{2}x - M = -EJ \frac{d^2 y}{dx^2}.$$

En l'intégrant deux fois, on obtient :

$$y = \frac{1}{EJ} \left[\frac{Mx^2}{2} - \frac{Xx^3}{12} \right] + C_1 x + C_2$$

Les deux constantes C_1 , C_2 se déterminent comme suit :

Pour $x = s - z$, $\frac{dy}{dx} = 0$, ce qui donne

$$C_1 = \frac{s-z}{EJ} \left[\frac{X(s-z)}{4} - M \right]$$

Pour $x=0$, $y=0$, ce qui donne

$$C_2=0.$$

De plus, comme pour $x=s-z$, $y=0$, on a :

$$M = \frac{N(s-z)}{3}$$

L'équation de y et celles de ses dérivées s'écrivent ensuite :

$$(1) \quad \begin{cases} y = \frac{Nx}{12EJ} \left[2(s-z)x - x^2 - (s-z)^2 \right] \\ \frac{dy}{dx} = \frac{N}{12EJ} \left[4(s-z)x - 3x^2 - (s-z)^2 \right] \\ \frac{d^2y}{dx^2} = \frac{N}{12EJ} \left[4(s-z) - 6x \right] \\ \frac{d^3y}{dx^3} = -\frac{N}{2EJ} \end{cases}$$

Nous avons maintenant dix inconnues à déterminer, savoir :

A_1 , A_2 , B_1 , B_2 , A_1' , A_2' , B_1' , B_2' , X et z .

Pour éviter de la confusion, nous écrivons toutes les équations de condition comme nous l'avons fait précédemment :

Au point A , on a

$$\frac{d^2y}{d\xi^2} = 0, \quad \frac{d^3y}{d\xi^3} = 0,$$

ce qui donne :

$$(i) \quad B_1 - B_2 = 0$$

$$(ii) \quad B_1 - A_1 + B_2 + A_2 = 0.$$

Au point C , $y \frac{dy}{d\xi}$, $\frac{d^2y}{d\xi^2}$ ont les mêmes valeurs pour les deux segments AC , CE , ce qui donne :

$$(iii) \quad [A_1 e^{\alpha} + A_2 e^{-\alpha}] \cos \alpha + [B_1 e^{\alpha} + B_2 e^{-\alpha}] \sin \alpha + \frac{2\eta}{c} = A_1' + A_2'$$

$$(iv) \quad [(A_1 + B_1) e^{\alpha} - (A_2 - B_2) e^{-\alpha}] \cos \alpha - [(A_1 - B_1) e^{\alpha} + (A_2 + B_2) e^{-\alpha}] \sin \alpha \\ = A_1' + B_1' - A_2' + B_2'$$

$$(v) \quad [B_1 e^{\alpha} - B_2 e^{-\alpha}] \cos \alpha - [A_1 e^{\alpha} - A_2 e^{-\alpha}] \sin \alpha = B_1' - B_2'$$

Au point E , on a :

$$[y]_{CE}=0, \quad \left[\frac{dy}{dx}\right]_{CE}=\left[\frac{dy}{dx}\right]_{EO}$$

$$-\frac{bCL}{4}\left[\frac{d^3y}{d\xi^3}\right]_{CE}=\frac{X}{2}, \quad -\frac{bCL^2}{4}\left[\frac{dy}{d\xi}\right]_{CE}=-M=-\frac{X(s-z)}{3},$$

ce qui donne :

$$(vi) \quad [A_1'e^{\xi} + A_2'e^{-\xi}]\cos\zeta + [B_1'e^{\xi} + B_2'e^{-\xi}]\sin\zeta = 0$$

$$(vii) \quad -A_1'e^{\xi}(\cos\zeta + \sin\zeta) + A_2'e^{-\xi}(\cos\zeta - \sin\zeta) + B_1'e^{\xi}(\cos\zeta - \sin\zeta) \\ + B_2'e^{-\xi}(\cos\zeta + \sin\zeta) = -\frac{2X}{bCL}$$

$$(viii) \quad A_1'e^{\xi}(\cos\zeta - \sin\zeta) - A_2'e^{-\xi}(\cos\zeta + \sin\zeta) + B_1'e^{\xi}(\cos\zeta + \sin\zeta) \\ + B_2'e^{-\xi}(\cos\zeta - \sin\zeta) = -\frac{(s-z)^2XL}{6EJ}$$

$$(ix) \quad A_1'e^{\xi}\sin\zeta - A_2'e^{-\xi}\sin\zeta - B_1'e^{\xi}\cos\zeta + B_2'e^{-\xi}\cos\zeta = \frac{-4X(s-z)}{3bCL^2}$$

Et enfin on a :

$$b\int_0^a p dx + b\int_0^z p dx = baq + \frac{X}{2}$$

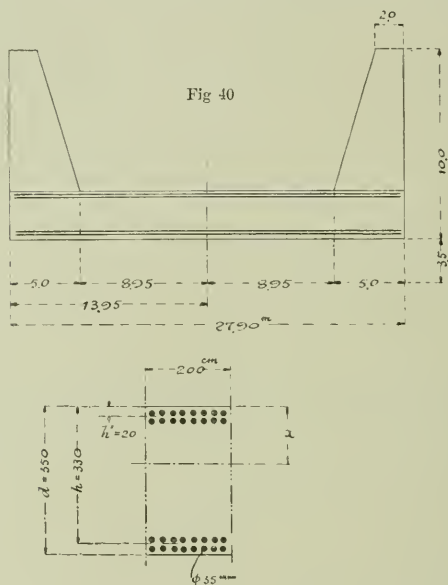
ce qui se traduit en :

$$(x) \quad A_1e^{\alpha}(\cos\alpha + \sin\alpha) + A_2e^{-\alpha}(-\cos\alpha + \sin\alpha) + B_1e^{\alpha}(\sin\alpha - \cos\alpha) \\ + B_2e^{-\alpha}(-\sin\alpha - \cos\alpha) - A_1' + A_2' + B_1' + B_2' = 0$$

En laissant de côté les deux équations (viii) (ix) nous dressons la table des coefficients :

A_1	A_2	$B_2=B_3$	A'_1	A'_2	B'_1	B'_2	
-1	1	2	0	0	0	0	0
$e^{\alpha} \cos \alpha$	$e^{-\alpha} \cos \alpha$	$(e^{\alpha} + e^{-\alpha}) \sin \alpha$	-1	-1	0	0	$-\frac{2q}{c}$
$e^{\alpha} (\cos \alpha - \sin \alpha)$	$-e^{-\alpha} (\cos \alpha + \sin \alpha)$	$(e^{\alpha} + e^{-\alpha}) \cos \alpha + (e^{\alpha} - e^{-\alpha}) \sin \alpha$	-1	1	-1	-1	0
$-e^{\alpha} \sin \alpha$	$e^{-\alpha} \sin \alpha$	$(e^{\alpha} - e^{-\alpha}) \cos \alpha$	0	0	-1	1	0
$e^{\alpha} (\cos \alpha + \sin \alpha)$	$e^{-\alpha} (-\cos \alpha + \sin \alpha)$	$-(e^{\alpha} + e^{-\alpha}) \cos \alpha + (e^{\alpha} - e^{-\alpha}) \sin \alpha$	-1	1	1	1	0
0	0	0	$e^{\alpha} \cos \alpha$	$e^{-\alpha} \cos \alpha$	$e^{\alpha} \sin \alpha$	$e^{-\alpha} \sin \alpha$	0
0	0	0	$-e^{\alpha} (\cos \alpha + \sin \alpha)$	$-e^{-\alpha} (\cos \alpha - \sin \alpha)$	$e^{\alpha} (\cos \alpha - \sin \alpha)$	$-e^{-\alpha} (\cos \alpha + \sin \alpha)$	$-\frac{2V}{bCL}$

2. Nous prenons comme exemple numérique, une forme de radoub dont la figure représente les dimensions principales.



$$\left\{ \begin{array}{l} C = 50 \frac{kg}{cm^3} \\ n = 15 \\ E = 143,200 \frac{kg}{cm^2} \\ x = 79.56 \text{ cm} \\ J = 127,000,000 \text{ cm}^4 \\ L = 347.30 \text{ cm} \\ a = \frac{50}{347.30} = 1.44 \text{ rad.} \\ \bar{a} = \frac{500}{347.30} = 2.58 \text{ rad.} \end{array} \right.$$

Nous calculons le radier :

- i. lorsqu'il repose sur le fond sans être retenu en aucun point
- ii. lorsqu'il est retenu au milieu.
- i. Nous nous servons de la table donnée* qui devient pour notre cas :

A_1	A_2	$B_1=B_2$	A_1'	A_2'	B_1'	B_2'
-1	1	2	0	0	0	0
0.530	0.031	4.415	-1	-1	0	0
-3.630	-0.266	4.526	-1	1	-1	-1
-4.180	0.235	0.519	0	0	-1	1
-4.730	-0.204	-3.364	1	-1	-1	-1
0	0	0	$e^{\zeta} \cos \zeta$	$e^{-\zeta} \cos \zeta$	$e^{\zeta} \sin \zeta$	$e^{-\zeta} \sin \zeta$
0	0	0	$-e (\cos \zeta + \sin \zeta)$	$e^{-\zeta} (\cos \zeta - \sin \zeta)$	$e^{\zeta} (\cos \zeta - \sin \zeta)$	$e^{-\zeta} (\cos \zeta + \sin \zeta)$

et de l'équation

* Page 239, Memoirs of the college of Engineering, Kyushu Imp. University, Vol. 1. No. 4. 1919.

$$A_1' e^{\zeta} (\cos \zeta - \sin \zeta) - A_2' e^{-\zeta} (\cos \zeta + \sin \zeta) + B_1' e^{\zeta} (\cos \zeta + \sin \zeta) \\ + B_2' e^{-\zeta} (\cos \zeta - \sin \zeta) + 2(\eta - \zeta) (-A_1' e^{\zeta} \sin \zeta + A_2' e^{-\zeta} \sin \zeta) = 0$$

On évalue :

$$\begin{cases} \zeta = 0.883 \text{ rad.} \\ z = 0.883 \times 347.30 = 306.80 \text{ cm} \end{cases}$$

Et ensuite on trouve :

$$\begin{cases} A_1 = -2.07 \times 10^{-4} q & A_1' = -550 \times 10^{-4} q \\ A_2 = 95.00 \times 10^{-4} q & A_2' = 190.00 \times 10^{-4} q \\ B_1 = B_2 = -48.60 \times 10^{-4} q & B_1' = -18.05 \times 10^{-4} q \\ & B_2' = -24.05 \times 10^{-4} q \end{cases}$$

On obtient les équations :

$$\begin{aligned} \text{pour la partie } AC \quad & \begin{cases} y = 10^{-4} q [(-1.035e^{\zeta} + 47.50e^{-\zeta}) \cos \zeta - 48.60 \cos \zeta \sin \zeta + 200] \\ M = -15070.00 q [-97.20 \sin \zeta \cos \zeta + (2.07e^{\zeta} + 95.00e^{-\zeta}) \sin \zeta] \\ Q = -43.40 q [-92.92 \sin \zeta \cos \zeta + (50.67e^{\zeta} - 143.60e^{-\zeta}) \sin \zeta] \end{cases} \\ \text{pour la partie } CE \quad & \begin{cases} y = \frac{10^{-4} q}{2} [(-5.50e^{\zeta} + 190.00e^{-\zeta}) \cos \zeta - (18.05e^{\zeta} + 24.05e^{-\zeta}) \sin \zeta] \\ M = -15070.00 q [(-18.05e^{\zeta} + 24.05e^{-\zeta}) \cos \zeta \\ \quad \quad \quad - (5.50e^{\zeta} - 190.00e^{-\zeta}) \sin \zeta] \\ Q = -43.40 q [(-12.55e^{\zeta} + 165.95e^{-\zeta}) \cos \zeta \\ \quad \quad \quad + (23.55e^{\zeta} - 204.05e^{-\zeta}) \sin \zeta] \end{cases} \\ \text{pour la partie } EO \quad & \begin{cases} y = -24.885 \times 10^{-4} q \zeta (3.394 - \zeta) \end{cases} \end{aligned}$$

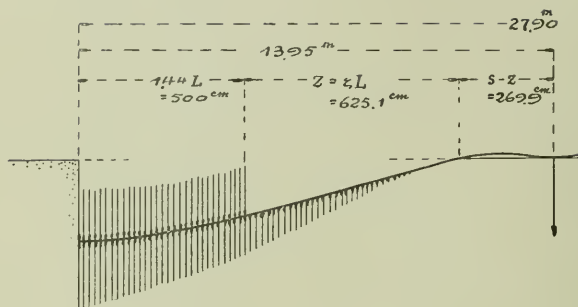
ii. La table des coefficients devient :

A_1	A_2	$B_1=B_2$	A_1'	A_2'	B_1'	B_2'	
-1	1	2	0	0	0	0	0
0.550	0.031	4.415	-1	-1	0	0	$-\frac{2\eta}{c}$
-3.630	-0.266	4.526	-1	1	-1	-1	0
-4.180	0.235	0.519	0	0	-1	1	0
4.730	0.204	3.364	-1	1	1	1	0
0	0	0	$e^{\lambda} \cos \zeta$	$e^{-\lambda} \cos \zeta$	$e^{\lambda} \sin \zeta$	$e^{-\lambda} \sin \zeta$	0
0	0	0	$-e^{\lambda} (\cos \zeta + \sin \zeta)$	$e^{-\lambda} (\cos \zeta - \sin \zeta)$	$e^{\lambda} (\cos \zeta + \sin \zeta)$	$e^{-\lambda} (\cos \zeta + \sin \zeta)$	$-\frac{2\lambda}{bCU}$

Au moyen de cette table on forme les expressions $A_1, A_2, \dots$ en fonction de ζ, λ et en les introduisant dans les deux équations (viii) (ix), on peut chercher à déterminer par tâtonnement les valeurs de $s-z$ et de λ :

$$\begin{cases} s-z=269.90 \text{ cm} \\ \lambda=4665q. \end{cases}$$

Fig. 41



On a donc :

$$\begin{cases} z = 895 - 269.90 \\ \quad = 625.10 \text{ cm} \\ \zeta = \frac{625.10}{347.30} = 1.8 \text{ rad.} \end{cases}$$

De ces valeurs on obtient :

$$\begin{cases} A_1 = -6.65 \times 10^{-4} q & A_1' = -0.052 \times 10^{-4} q \\ A_2 = 87.10 \times 10^{-4} q & A_2' = 191.00 \times 10^{-4} q \\ B_1 = B_2 = -46.85 \times 10^{-4} q & B_1' = 1.84 \times 10^{-4} q \\ & B_1' = -22.12 \times 10^{-4} q \end{cases}$$

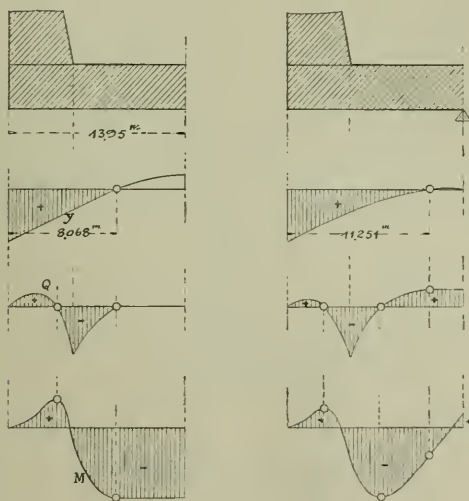
et l'on a :

$$\begin{cases} y = 10^{-4} q [(-3.33e^{\zeta} + 43.50e^{-\zeta}) \cos \zeta - 46.85 \zeta \cos \zeta \sin \zeta + 200] \\ M = -15070.00 [-93.70 \zeta \sin \zeta \cos \zeta + (6.65e^{\zeta} + 87.10e^{-\zeta}) \sin \zeta] q \\ Q = -43.40 q [(-40.20e^{\zeta} + 40.25e^{-\zeta}) \cos \zeta + (53.50e^{\zeta} - 133.95e^{-\zeta}) \sin \zeta] \end{cases}$$

$$\left\{ \begin{array}{l} y = \frac{10^{-4}q}{2} \left[(-0.052e^{\xi} + 191.00e^{-\xi})\cos\xi + (1.84e^{\xi} - 22.12e^{-\xi})\sin\xi \right] \\ M = 15070.00q \left[(1.84e^{\xi} + 22.12e^{-\xi})\cos\xi + (0.052e^{\xi} + 191.00e^{-\xi})\sin\xi \right] \\ Q = 43.40q \left[(1.892e^{\xi} + 168.88e^{-\xi})\cos\xi - (1.788e^{\xi} + 213.120e^{-\xi})\sin\xi \right] \end{array} \right.$$

$$\left\{ \begin{array}{l} y = 2.140 \times 10^{-4} [269.90(2x - 269.90) - x^2] x q \\ M = -388.60q (1079.60 - 6x) \\ Q = 2333q \end{array} \right.$$

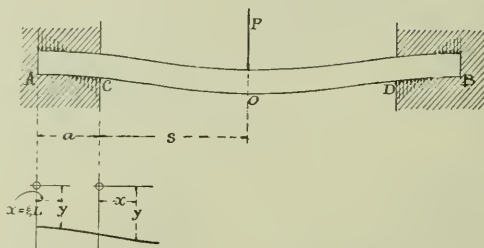
Fig. 42



IX. Poutre encastrée dans un milieu élastique.

1. Soit AB une poutre encastrée à ses deux extrémités, supportant, en son milieu, un poids P .

Fig. 43.



Pour la partie encastrée AC , nous supposons l'équation

$$y = \frac{1}{2} \left[(A_1 e^{\xi} + A_2 e^{-\xi}) \cos \xi + (B_1 e^{\xi} + B_2 e^{-\xi}) \sin \xi \right]$$

Quant à la partie CO , nous avons l'équation différentielle :

$$EJ \frac{d^4 y}{dx^4} = 0$$

ou

$$\frac{d^4 y}{dx^4} = 0,$$

ce qui nous donne comme l'intégrale :

$$y = A_1' x^3 + A_1' x^2 + B_1' x + B_2'.$$

Et l'on a :

$$\begin{cases} M = -EJ \frac{d^2 y}{dx^2} = -2EJ(3A_1' x + A_2') \\ Q = -EJ \frac{d^3 y}{dx^3} = -6EJA_1' \end{cases}$$

Nous avons ici les huit constantes $A_1, A_2, B_1, B_2, A_1', A_2', B_1', B_2'$ à déterminer. Les équations de condition se forment :

A l'extrémité A ,

$$\frac{d^2 y}{dz^2} = 0, \quad \frac{d^3 y}{dz^3} = 0,$$

ce qui donne :

$$(i) \quad B_1 - B_2 = 0$$

$$(ii) \quad B_1 - A_1 + B_2 + A_2 = 0$$

Au point C ,

$$y = 0, \quad \frac{dy}{dx} = \frac{dy'}{dx'},$$

$$-\frac{bCL^2}{4} \left[\frac{d^2 y}{dz^2} \right]_{z=x} = -EJ \left[\frac{d^2 y}{dx} \right]_{x=0},$$

$$-\frac{bCL}{4} \left[\frac{d^3 y}{dz^3} \right]_{z=x} = Q_x = 0,$$

ce qui donne :

$$(iii) \quad A_1 e^{\alpha} \cos \alpha + A_2 e^{-\alpha} \cos \alpha + B_1 e^{-\alpha} \sin \alpha + B_2 e^{-\alpha} \sin \alpha = 2B_2'$$

$$(iv) \quad \frac{1}{2L} \left[A_1 e^{\alpha} (\cos \alpha - \sin \alpha) - A_2 e^{-\alpha} (\cos \alpha + \sin \alpha) + B_1 e^{\alpha} (\cos \alpha + \sin \alpha) \right. \\ \left. + B_2 e^{-\alpha} (\cos \alpha - \sin \alpha) \right] = B_1'$$

$$(v) \quad B_1 e^{\alpha} \cos \alpha - B_2 e^{-\alpha} \cos \alpha - A_1 e^{\alpha} \sin \alpha + A_2 e^{-\alpha} \sin \alpha = \frac{8EJ}{bCL^2} A_2'$$

$$(vi) \quad -A_1 e^{\alpha} (\cos \alpha + \sin \alpha) + A_2 e^{-\alpha} (\cos \alpha - \sin \alpha) + B_1 e^{\alpha} (\cos \alpha - \sin \alpha) \\ + B_2 e^{-\alpha} (\cos \alpha + \sin \alpha) = \frac{24EJ}{bCL} A_1'$$

Au centre

$$\frac{dy}{dx} = 0, \quad Q = \frac{P}{2},$$

ce qui donne :

$$(vii) \quad 3A_1' s^2 + 2A_2' s + B_1' = 0$$

$$(viii) \quad -6EJA_1' = \frac{P}{2}.$$

De ces huit equations, on dresse la table :

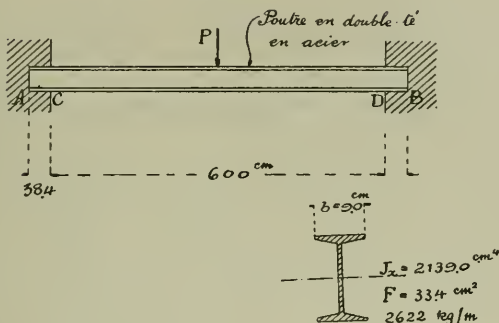
A_1	A_2	$B_1 = B_2$	A'_1	A'_2	B'_1	B'_2
-1	1	2	0	0	0	0
$e^{\alpha} \cos \alpha$	$e^{-\alpha} \cos \alpha$	$(e^{\alpha} + e^{-\alpha}) \sin \alpha$	0	0	0	-2
$e^{\alpha} (\cos \alpha - \sin \alpha)$	$-e^{-\alpha} (\cos \alpha + \sin \alpha)$	$(e^{\alpha} + e^{-\alpha}) \cos \alpha + (e^{\alpha} - e^{-\alpha}) \sin \alpha$	0	0	-2L	0
$e^{\alpha} \sin \alpha$	$e^{-\alpha} \sin \alpha$	$(e^{\alpha} - e^{-\alpha}) \cos \alpha$	0	-2L ²	0	0
$e^{\alpha} (\cos \alpha + \sin \alpha)$	$e^{-\alpha} (\cos \alpha - \sin \alpha)$	$(e^{\alpha} + e^{-\alpha}) \cos \alpha - (e^{\alpha} - e^{-\alpha}) \sin \alpha$	-6L ³	0	0	0
0	0	0	3s ²	2s	1	0
0	0	0	$-\frac{3}{2} bGL^4$	0	0	$\frac{P}{2}$

Les déterminants se développent :

$$\begin{aligned}
 (1) \quad & \begin{cases} \Delta = 24bGL^3[2s(\sin^2 \alpha - \sin^2 \alpha) + L(\sin 2\alpha + \sin 2\alpha)] \\ \Delta_{A_1} = 24PL^3[L^2\{\cos \alpha(\cos \alpha + \sin \alpha) + e^{-\alpha} \cos \alpha\} + 2sL\{\sin \alpha(\cos \alpha - \sin \alpha) + e^{-\alpha} \sin \alpha\}] \\ \Delta_{A_2} = 24PL^3[L^2\{\cos \alpha(\cos \alpha - \sin \alpha) + e^{\alpha} \cos \alpha\} + 2sL\{\sin \alpha(\cos \alpha + \sin \alpha) + e^{\alpha} \sin \alpha\}] \\ \Delta_{B_1} = \Delta_{B_2} = 24PL^3[L^2\{\cos \alpha \sin \alpha - \sin \alpha \cos \alpha\} + 2sL\{\sin \alpha \sin \alpha + s^2(\sin \alpha \cos \alpha + \cos \alpha \sin \alpha)\}] \end{cases}
 \end{aligned}$$

Appliquons les formules à un exemple numérique :

Fig. 44



$$\begin{cases} C = 500 \frac{\text{kg}}{\text{cm}^3} \\ L = 44.649 \text{ cm} \\ a = \frac{38.4}{44.649} = 0.86 \text{ rad.} \end{cases}$$

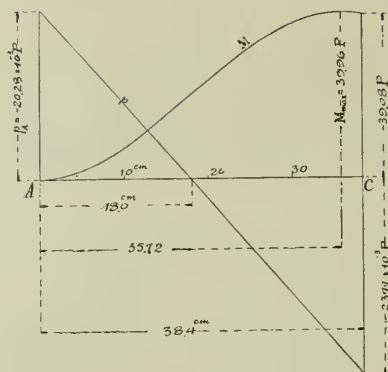
On évalue :

$$\begin{cases} A_1 = 0.938 \times 10^{-5} P \\ A_2 = -9.050 \times 10^{-5} P \\ B_1 = B_2 = 5.000 \times 10^{-5} P \end{cases}$$

et l'on obtient :

$$\begin{cases} y = \frac{10^{-5} P}{2} [0.938 e^{\xi} \cos \xi - 9.050 e^{-\xi} \cos \xi + 10.000 \cos \xi \sin \xi] \\ M = -22.420 P [10.000 \sin \xi \cos \xi - 0.938 e^{\xi} \sin \xi - 9.050 e^{-\xi} \sin \xi] \end{cases}$$

Fig. 45



On remarque sur la figure que le maximum du moment fléchissant n'a pas lieu au point d'encastrement, mais il se trouve un peu plus loin dans l'intérieur du milieu. Pour notre cas, nous avons :

$$M_{max} = -39.960P \text{ (Correspondant à } x = 35.72 \text{ cm)}$$

En faisant $\xi = a$ dans l'équation de M , on a :

$$(2) \quad M_0 = -\frac{P[s^2(\Xi \sin^2 a - \sin^2 a) - L^2(\Xi \sin a + \sin^2 a)]}{2[2s(\Xi \sin^2 a - \sin^2 a) + L(\Xi \sin 2a + \sin 2a)]}$$

C	M_C	C	M_C
10	- 1.783 P	1,000	-47.350 P
50	-10.700 „	10,000	-61.800 „
100	-17.290 „	100,000	-62.250 „
300	-30.700 „	∞	-75.000 „
500	-39.080 „		

On voit que :

$$\lim_{c \rightarrow \infty} M_c = -\frac{Ps}{4}$$

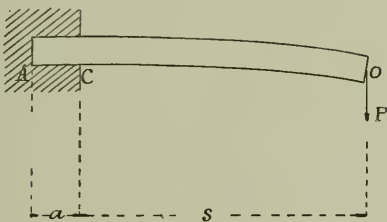
2. Donnons les formules pour la poutre encastrée d'un bout.

Ici les deux dernières équations de condition seulement se modifient :

Au point O ,

$$\frac{d^2y}{dx^2} = 0, \quad Q = P,$$

Fig. 46



ce qui donne :

$$(vii) \quad 6A_1's + 2A_2' = 0$$

$$(viii) \quad -6EJA_1' = P.$$

La table des coefficients devient donc :

A_1	A_2	$B_1=B_2$	A'_1	A'_2	B'_1	B'_2	
-1	1	2	0	0	0	0	0
$e^\alpha \cos \alpha$	$e^{-\alpha} \cos \alpha$	$(e^\alpha + e^{-\alpha}) \sin \alpha$	0	0	0	-2	0
$e^\alpha (\cos \alpha - \sin \alpha)$	$-e^{-\alpha} (\cos \alpha + \sin \alpha)$	$(e^\alpha + e^{-\alpha}) \cos \alpha + (e^\alpha - e^{-\alpha}) \sin \alpha$	0	0	-2L	0	0
$-e^\alpha \sin \alpha$	$e^{-\alpha} \sin \alpha$	$(e^\alpha - e^{-\alpha}) \cos \alpha$	0	-2L ²	0	0	0
$-e^\alpha (\cos \alpha - \sin \alpha)$	$e^{-\alpha} (\cos \alpha + \sin \alpha)$	$(e^\alpha + e^{-\alpha}) \cos \alpha - (e^\alpha - e^{-\alpha}) \sin \alpha$	-6L ³	0	0	0	0
0	0	0	6s	2	0	0	0
0	0	0	$-\frac{3}{2}bCL^4$	0	0	0	P

Et l'on a :

$$\begin{cases}
 \Delta = 48bCL^4(\sin^2 \alpha - \sin^2 a) \\
 \Delta A_1 = 96PL^3[L(\sin \alpha \cos \alpha - e^{-\alpha} \sin \alpha) + s\{\sin \alpha (\cos \alpha - \sin \alpha) + e^{-\alpha} \sin \alpha\}] \\
 \Delta A_2 = 96PL^3[L(\sin \alpha \cos \alpha - e^\alpha \sin \alpha) - s\{\sin \alpha (\cos \alpha + \sin \alpha) + e^\alpha \sin \alpha\}] \\
 \Delta B_1 = \Delta B_2 = 96PL^3[L \sin \alpha \sin a + s(\sin \alpha \cos \alpha \cos \alpha \sin a)]
 \end{cases}
 \quad (3)$$

Miscellaneous Notes on Impact Tests.

By

Prof. Akimasa Ono, *Kogakuhakushi.*

(Received July 5, 1919)

Introductory.

Toughness of materials against impact (the contrary property is fragility) can hardly escape the attention of engineers in selecting proper materials of construction. So there are now several ways for testing the said property, and Charpy's pendulum machine is too well known to need description here. The present article is the outcome of several impact tests chiefly performed in the 75 *mkg.* pendulum machine constructed by Messrs. Mohr and Federhaff of Mannheim, with the exception of two series of tests, in which Prof. Matsumura's repeated impact testing machine¹⁾ was used.

Most of the microphotographs reproduced in the present article were taken with Le Chatelier's microscope and camera in the metallographical laboratory under the charge of Prof. Kaneko. All the experiments including the mechanical, metallographical, and chemical tests were performed by Mr. Omura, assistant, and Mr. Tanaka was also engaged in some part of the test.

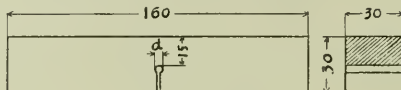
1) Journal of the Society of Mechanical Engineers, Tokyo, 1918.

I. Influence of the Diameter of Holes.

The form of the test pieces taken in the present experiment was a bar, 160 mm. in length, with a square section of 30 mm. side, Fig. 1.

C, G, H, K, L, O, P.

Fig. 1.



A round hole was drilled at the centre of the length, communicating with the outside by a saw cut, the narrowest portion of the piece being of rectangular section 15 mm. $\times$ 30 mm. The diameter of the hole was varied in certain range in order to clear up the influence of the hole diameter on the specific work in fracture per square centimeter of breaking section, as usually calculated in this kind of test.

A similar subject has already been studied by Ehrensberger,¹⁾ but the diameters taken in his experiment being 4 mm. in one kind and 6 mm. in the other, the range was too wide for my purpose. So in the present case the diameter has been varied in a more narrow range. This was evidently not advantageous to determine the general feature of the problem, as the irregularity of materials among other things would be more sensible in the case of the smaller range than the wider. Still the mean rate of the change of work in fracture appeared to be necessarily determined from the results of a narrow range in order that the

1) Zeitschrift d. Vereines deutscher Ingenieure, 1907, p. 1974.

rate obtained should give the approximate value in the neighborhood of a definite diameter fixed as the standard.

Series C.

The test pieces for the statical and impact tests were taken from an ingot iron bar of square section, Fig. 2, annealed at a little higher temperature than 700°C .

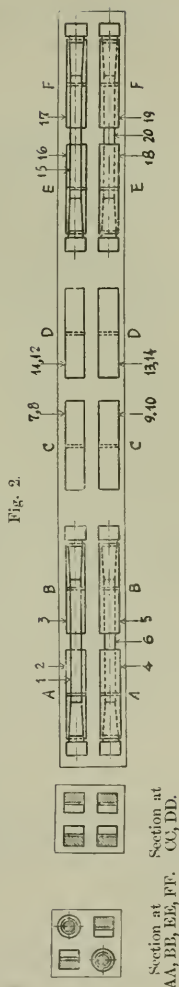
The chemical composition of the material was reported as follows¹⁾:

<i>C</i>	<i>Si</i>	<i>Mn</i>	<i>P</i>	<i>S</i>
0.22	0.008	0.45	0.025	0.027

The results of the statical test, Table I, show that the behavior of the material was rather uniform, while those of the impact test, Table II, are also tolerably regular except in the case of the smallest hole diameter, 3.80 mm., the average work in fracture being greater than those for the diameters 4.00 and 4.20 mm.

Notwithstanding this defect, it appears desirable to deduce a relation between the work in fracture *A* and the hole diameter *d*. Thus assuming the linear law in the present range of *d*, we obtain

$$A = 0.253 d + 0.701,$$



1) Due to the Imperial Steel Works.

where A is measured in *mkg.* per cm^2 , while d is expressed in *mm.*

Series K.

The preparation of the test pieces was performed in a similar way to Series *C*, the material being ingot iron with an average carbon content of 0.196%.

According to the results of the statical test, Table III, the present material had greater breaking extension and area contraction than the material for Series *C*. Further the statical work in fracture is found to be higher than the foregoing case, though the difference is not so conspicuous as the results of the impact test, Table IV, compared with those of Table II. One impact test piece with the hole diameter 4.40 *mm.* and two pieces with the hole diameter 4.60 *mm.* were not broken.

It is to be seen from Table IV that the impact test pieces Nos. 2 and 17 (hole diameters 4.00 *mm.*) showed rather great difference in the work in fracture. So Brinell's hardness test was performed on these pieces with the result that no difference was observed, the mean value determined from two indentations in each piece being equal to 120.

The linear law for A and d runs as follows:

$$A = 1.770 d + 4.768,$$

the test on the pieces with hole diameters 4.40 and 4.60 *mm.* being wholly disregarded in the calculation of the numerical constants.

Series L.

The test pieces of the present and the following two series

were taken from a long bar with a square section of 38 mm. side, and they were numbered 1, 2, 3, ... in succession, the extreme small portions of the original bar being rejected.

The present material was low carbon steel, the percentage of carbon being 0.042 on an average. The specimens were all annealed in an electric furnace at 900°C. for half an hour and were cooled down in the furnace.

The first and the last specimens were tensile test pieces on which the statical test was made with the results as shown in Table V.

From the results of the impact test, Table VI, we have

$$A = 0.109 d + 0.917.$$

Series O.

The test pieces were prepared in a similar way to Series L, except that the statical test pieces were not taken in the present and the following cases. The carbon content of the present material is 0.081%.

The results of the impact test, Table VII, may be expressed as follows:

$$A = 0.405 d + 1.618.$$

Series P.

The test pieces were taken in a similar way to Series O, the carbon content of the present material being 0.159% on an average.

As Table VIII shows, the test pieces Nos. 2 and 3 were not broken owing to the insufficiency of the energy of the blow. In testing the three specimens No. 1 to No. 3, the initial angle

of pendulum was taken at 45° , while in all the others the angle was 60° . Judging from the fairly uniform nature of the specimens, except Nos. 2 and 3 above remarked, it appears that the work in fracture is not independent of the velocity of the impinging hammer. The relation between A and d is as follows:

$$A = 0.218 d + 1.441$$

The foregoing results of Series C , K , L , O , and P are graphically shown in Fig. 3. Now, in accordance with the linear relation $A = ad + b$, the variation Δd results in the corresponding change of A thus

$$\Delta A = a \Delta d,$$

and therefore

$$\begin{aligned} \frac{\Delta A}{A} &= \frac{a \Delta d}{ad + b} \\ &= \frac{\Delta d}{d + \frac{b}{a}}, \end{aligned}$$

where the constants a and b depend on the nature of material.

So far as the above series of the tests are concerned, the values of the ratio $\frac{b}{a}$ are as follows:

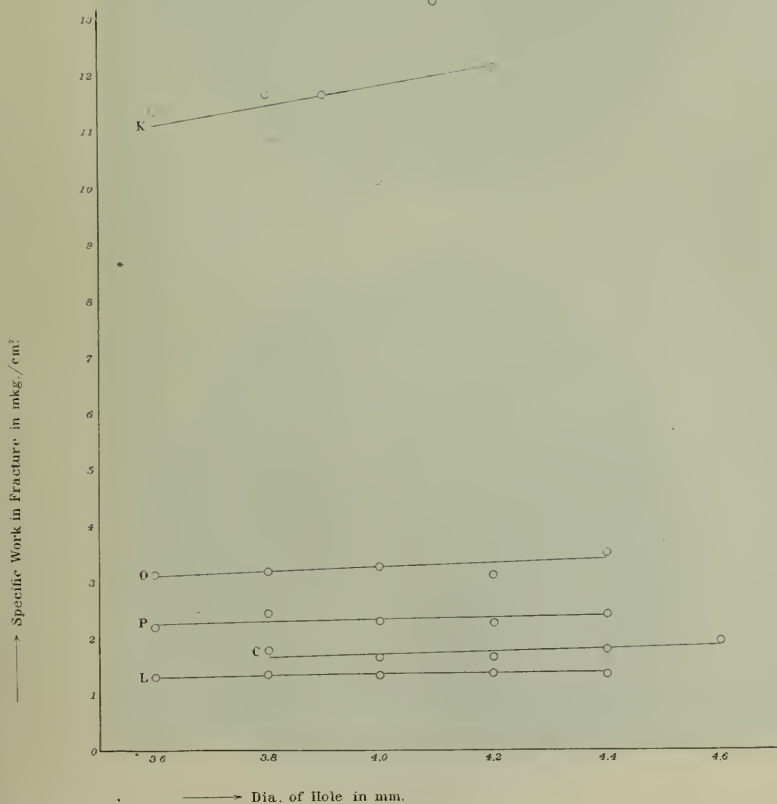
Series	C	K	L	O	P
$\frac{b}{a}$	2.77	2.69	8.41	4.00	6.61

The experimental results being very probably influenced by the lack of homogeneity of the material, these values of $\frac{b}{a}$ can not claim high accuracy, but they appear to be greatly dependent on the property of the material.

Here let us refer to Ehrensberger's report (loc. cit), from which the following mean values of the specific work in fracture are calculated.

94^a

Fig. 3.



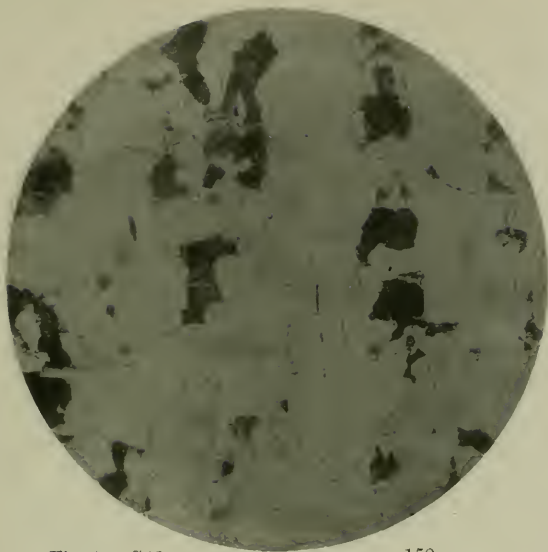


Fig. 4. C 11.

× 150

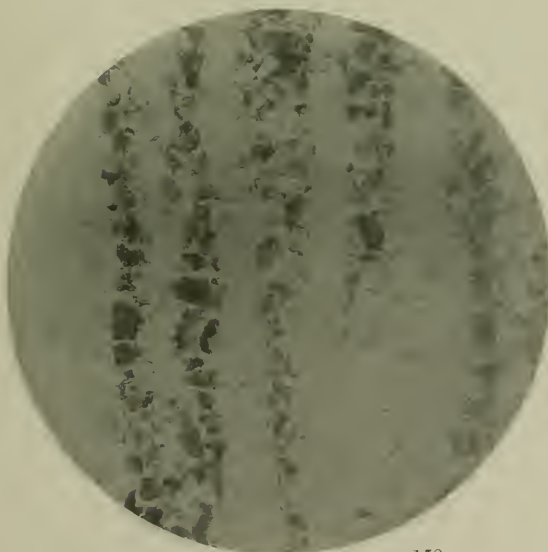


Fig. 5. K 11.

× 150

Material	$\overbrace{A}^{A \text{ mkg./cm}^2.}$		a	b	$\frac{b}{a}$
	$d=4 \text{ mm.}$	$d=6 \text{ mm.}$			
Ingot iron.....	33.8	33.1	2.15	25.2	11.7
Carbon steel	17.4	22.2	2.4	7.8	3.3
Nickel chrome steel...	26.2	29.7	1.75	19.2	11.0

If we assume the linear relation between A and d for the above range of d , the values of the constants are as shown in the above table. Thus except carbon steel $\frac{b}{a}$ is greater than in the author's tests.

Now, the smaller the ratio $\frac{b}{a}$, the greater $\frac{JA}{A}$ for definite values of d and JA , i.e. the more sensitive the material to the variation of diameter. For the present we take small value of $\frac{b}{a}$ to make $\frac{JA}{A}$ great. Thus with $\frac{b}{a}=2.7$ and $d=4 \text{ mm.}$,

$$\frac{JA}{A} = \frac{JA}{6.7}.$$

To drill a hole 4 mm. in diameter, it is not difficult to make the diameter remain within the limits of $4.0 \pm 0.1 \text{ mm.}$ Then

$$\frac{JA}{A} = \pm \frac{0.1}{6.7} \quad \text{or} \quad \pm 1.5\%.$$

Such a deviation is not considerable compared with the local inhomogeneity met with in the present kinds of materials.

The microphotographs, Figs. 4 and 5, show the structures of the specimens C and K . The latter figure represents the banding existing in the material, but when we consider the results of the impact tests, it appears that the banding is not necessarily the source of weakness, but the irregularity of toughness should be observed, cp. Fig. 3. These points will be con-

sidered more fully in the next section with reference to other examples.

Table I.

Static Test—Series C.

May 20 and 21, 1915.

Room Temperature 17° to 21°C.

No. of Test Piece	Yielding Point <i>kg./cm.²</i>	Tensile Strength <i>kg./cm.²</i>	Breaking Extension %	Area Contraction %	Work in Fracture <i>mkg./cm.³</i>
1	2065 & 1855	3817	26.4	51.0	8.94
6	2390 & 2041	3824	26.0	47.5	8.88
15	2066	3813	28.4	51.9	9.81
20	2224 & 1904	3730	28.0	55.8	9.28

Table II.

Impact Test—Series C.

May 21 to 26, 1915.

Room Temperature 20.7° to 21.6°C.

No. of Test Piece	Dia. of Hole <i>mm.</i>	Total Work in Fracture <i>mkg.</i>	Specific Work in Fracture <i>mkg./cm.²</i>	Do. mean <i>mkg./cm.²</i>
5	3.80	8.12	1.80	1.77
12		7.82	1.74	
13		7.97	1.77	
2	4.00	6.91	1.54	1.65
8		7.36	1.63	
9		7.97	1.77	
17		7.52	1.67	

No. of Test Piece	Dia. of Hole mm.	Total Work in Fracture mkg.	Specific Work in Fracture mkg./cm. ²	Do. mean mkg./cm. ²
7	4.20	7.97	1.77	1.67
11		7.36	1.63	
14		7.21	1.60	
3	4.40	8.12	1.80	1.80
16		7.82	1.74	
18		8.42	1.87	
4	4.60	8.27	1.84	1.94
10		7.97	1.77	
19		9.92	2.20	

Table III.

Statical Test—Series K.

June 8 and 9, 1915.

Room Temperature 20.1° to 22.7° C.

No. of Test Piece	Yielding Point kg./cm. ²	Tensile Strength kg./cm. ²	Breaking Extension %	Area Contraction %	Work in Fracture mkg./cm. ³
1	2231 1912	3985	30.5	64.5	11.1
6	2072	4017	30.3	64.0	11.1
15	1912 1721	3897	32.6	69.7	11.4
20	2231 1742	3904	29.3	69.7	10.3

Table IV.

Impact Test—Series K.

June 10, 1915.

Room Temperature 21.5° to 21.6°C.

No. of Test Piece	Dia. of Hole mm.	Total Work in Fracture mkg.	Specific Work in Fracture mkg./cm. ²	Do. mean mkg./cm. ²
3	3.60	48.28	10.73	11.37
5		54.04	12.01	
12	3.80	51.39	11.42	11.66
13		53.56	11.90	
8	3.90	55.74	12.39	11.69
9		49.44	10.99	
2	4.00	35.64	7.92	10.12
17		55.42	12.32	
7	4.10	60.66	13.48	13.38
10		59.76	13.28	
11	4.20	53.95	11.99	12.16
14		55.46	12.32	
16	4.40	65.43	14.54	not broken
18		>75.00	>16.67	
4	4.60	>56.88	>12.64	not broken
19		>75.00	>16.67	

Table V.

Statical Test—Series L.

April 5, 1916.

Room Temperature 15.5°C.

No. of Test Piece	Yielding Point kg./cm. ²	Tensile Strength kg./cm. ²	Breaking Extension %	Area Contraction %	Work in Fracture mkg./cm. ³
1	3181 2621	4098	31.5	68	9.92
27	2864 2513	4162	29.5	67	9.30

Table VI.

Impact Test—Series L.

April 6, 1916.

Room Temperature 16.5 to 16.6° C.

No. of Test Piece	Dia. of Hole <i>mm.</i>	Total Work in Fracture <i>mkg.</i>	Specific Work in Fracture <i>mkg./cm.²</i>	Do. mean <i>mkg./cm.²</i>
2	3.6	5.85	1.30	1.290
7		5.63	1.25	
12		5.85	1.30	
17		5.85	1.30	
22		5.85	1.30	
3	3.8	6.60	1.465	1.352
8		5.85	1.30	
13		5.74	1.275	
18		6.38	1.42	
23		5.85	1.30	
4	4.0	5.96	1.325	1.348
9		6.28	1.395	
14		5.96	1.325	
19		5.85	1.30	
24		6.28	1.395	
5	4.2	6.06	1.345	1.400
10		6.38	1.42	
15		5.96	1.325	
20		6.70	1.49	
25		6.38	1.42	
6	4.4	6.06	1.345	1.375
11		5.96	1.325	
16		6.49	1.44	
21		6.28	1.395	
26		6.17	1.37	

Table VII.

Impact Test—Series O.

Dec. 20, 1916.

Room Temperature 16.0° to 17.8°C.

No. of Test Piece	Dia. of Hole <i>mm.</i>	Total Work in Fracture <i>mkg.</i>	Specific Work in Fracture <i>mkg./cm.²</i>	Do. mean <i>mkg./cm.²</i>
1	3.6	14.06	3.13	3.10
6		14.88	3.31	
11		14.06	3.13	
16		12.65	2.81	
2	3.8	14.42	3.21	3.16
7		14.06	3.13	
12		14.79	3.29	
17		13.50	3.00	
3	4.0	13.96	3.11	3.29
8		14.24	3.17	
13		16.90	3.76	
18		13.96	3.11	
4	4.2	12.46	2.77	3.11
9		16.29	3.62	
14		14.33	3.19	
19		12.84	2.85	
5	4.4	16.55	3.68	3.53
10		16.03	3.57	
15		15.50	3.45	
20		15.41	3.43	

Table VIII.

Impact Test—Series P.

Dec. 21, 1916.

Room Temperature 17.5° to 18.5°C.

No. of Test Piece	Dia. of Hole mm.	Total Work in Fracture mkg.	Specific Work in Fracture mkg./cm. ²	Do. mean mkg./cm. ²
1	3.6	8.28	1.84	2.17
6		10.26	2.28	
11		10.39	2.31	
16		10.12	2.25	
2	3.8	>33.32	>7.41	Not broken
7		10.39	2.31	2.41
12		10.65	2.37	
17		11.45	2.55	
3	4.0	>33.32	>7.41	Not broken
8		10.65	2.37	2.30
13		10.52	2.34	
18		9.85	2.19	
4	4.2	9.72	2.16	2.27
9		11.19	2.49	
14		10.26	2.28	
19		9.59	2.13	
5	4.4	10.39	2.31	2.44
10		9.99	2.22	
15		11.59	2.58	
20		11.85	2.64	

II. Effect of Banding in Structure.

There are two series of impact tests, *G* and *H*, which were performed in the similar way as Series *C* and *K*, but of which the report was reserved for the present section, as the results appeared to be worth special consideration. The test pieces, Fig. 1, for each of these series were prepared from a bar as shown in Fig. 2, and the diameter of holes varied from 3.60 mm. to 4.60 mm. The results of the statical and impact tests are shown in Table IX to XII. As to the statical tests, no special remark is necessary, but the impact test showed such an irregularity that no definite law could be deduced especially in the case of Series *H*. In a particular case the work in fracture by impact was more than 4 times that of another test piece with the same hole diameter.

To make the cause of the divergence clear by submitting the material to a further investigation, Brinell's hardness test was performed on some particular pieces. The hardness number was determined at both ends of the impact piece. The mean values are shown in the following table.

Series G.

No. of Test Piece	Hardness
3	132
5	132
18	135
16	133

Series H.

No. of Test Piece	Hardness
3	120
5	133
10	120
7	133
19	129
4	131

The test pieces with the same hole diameter are arranged in pairs in the above table, and that which had greater toughness stands above the smaller one in each group. In the case of Series *G* no definite conclusion can be drawn, but in the case of Series *H* the hardness and toughness are in the reversed order, i.e. the harder material is weaker for impact.

Next the metallographical investigation was made, and some of the microphotographs are reproduced in Figs. 6 to 13.¹⁾

Fig. 6, test piece *G* 16, longitudinal section.

Fig. 7, „ „ *G* 16, cross section.

Fig. 8, „ „ *G* 18, longitudinal section.

Fig. 9, „ „ *G* 18, cross section.

Fig. 10, „ „ *H* 4, longitudinal section.

Fig. 11, „ „ *H* 4, cross section.

Fig. 12, „ „ *H* 19, longitudinal section.

Fig. 13, „ „ *H* 19, cross section.

1) The number added to the test piece mark in the photographs refers to the metallographical specimen.

The structure of the specimens of both series was found very irregular, and the banding was very marked, as shown in the longitudinal sections. Such structures may foretell the presence of a comparatively large amount of phosphorus.¹⁾

The chemical compositions of the present materials are given below.

	<i>C</i>	<i>Si</i> trace	<i>Mn</i>	<i>P</i>	<i>S</i>
<i>G</i>	0.302	<0.01	0.694	<u>0.053</u>	0.029
<i>H</i>	0.223	trace <0.01	0.630	<u>0.067</u>	0.038

The banding as caused by phosphorus is a symptom of inferior diffusibility, and it is natural that there are local irregularities in a piece of steel. So considerable divergence may occur in the results of impact tests owing to the inhomogeneity of regions surrounding notch holes.²⁾ The above statement is right when we consider the circumstance that the material *H* had higher *P* content and showed greater fluctuation of toughness than the other series *G*.

Here it is to be observed that banding itself does not mean weakness, as may be understood from the fact that there were some specimens which required a moderate amount of impact energy among the various fragile ones. Cp. also the remark at the end of the preceding section. But owing to the irregularity many pieces stood a very small amount of energy.

Whether steel bars with phosphorus banding should be rejected or not, is a matter of practical importance. Stead³⁾

1) Stead, the Journal of the Iron and Steel Institute, 1915, I, p. 140.

2) Rosenhain: Physical Metallurgy, p. 237.

3) Loc. cit. p. 167.

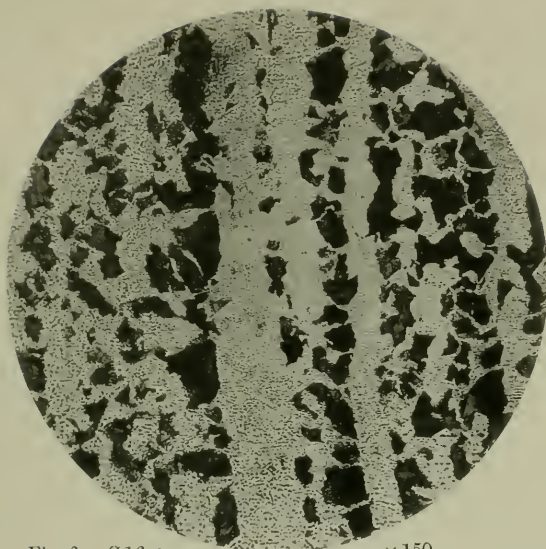


Fig. 6. *G* 16, 2.

× 150

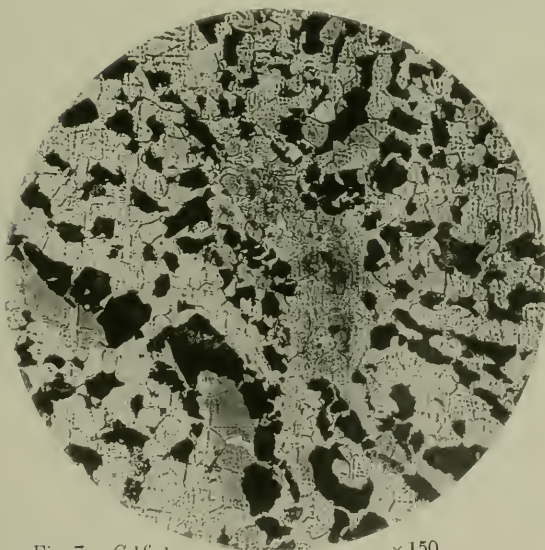
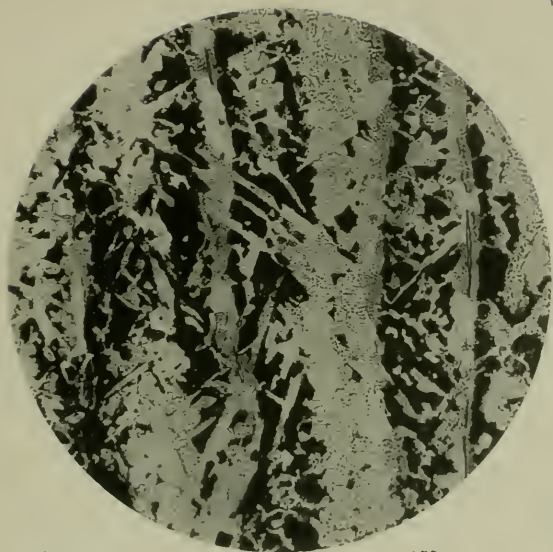
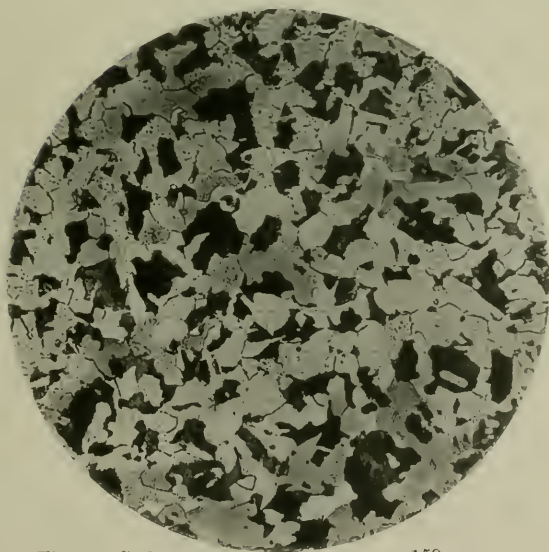


Fig. 7. *G* 16, 1.

× 150

Fig. 8. *G* 18. 2.

× 150

Fig. 9. *G* 18. 1.

× 150

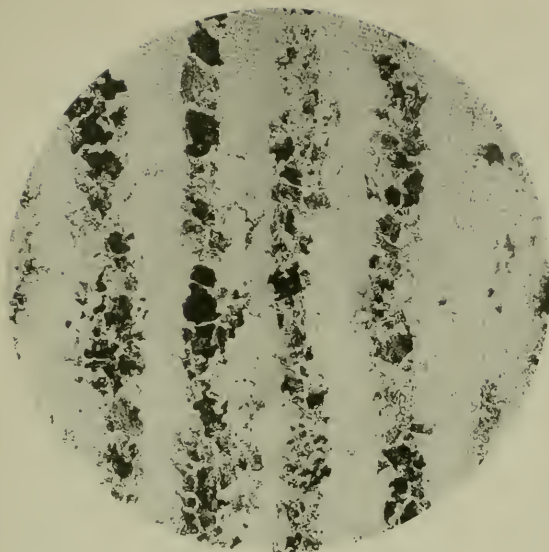


Fig. 10. $H4,3$.

$\times 150$

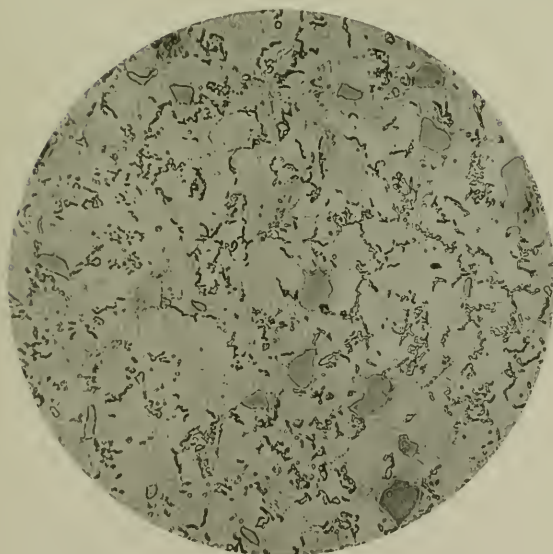


Fig. 11. $H4,4$.

$\times 150$

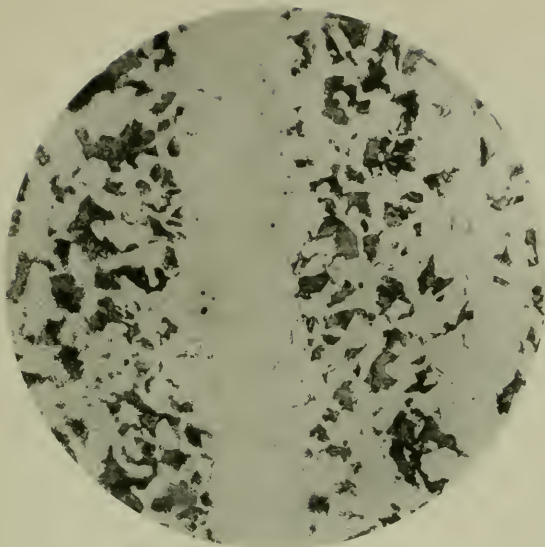


Fig. 12. *H* 19, 3.

× 150

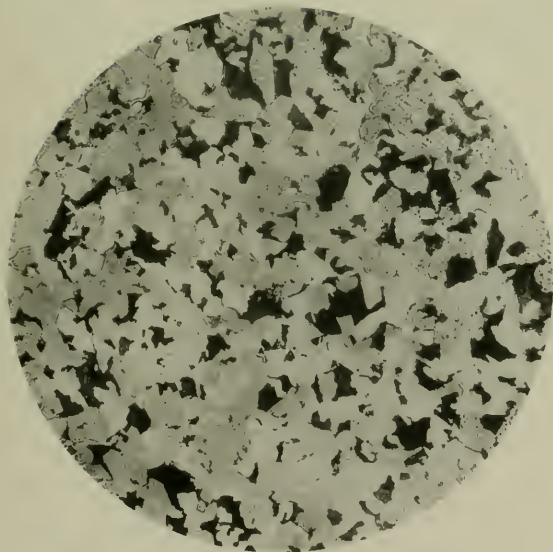


Fig. 13. *H* 19, 4.

× 150

mentions the belief that they are not dangerous unless a large amount of slag inclusions is present. This judgement remains at the discretion of engineers, who may find, it is hoped, some guide in the above series of tests.

Table IX.

Statical Test—Series G.

May 29 to June 1, 1915.

Room Temperature 21 to 25°C.

No. of Test Piece	Yielding Point <i>kg./cm.²</i>	Tensile Strength <i>kg./cm.²</i>	Breaking Extension %	Area Contraction %	Work in Fracture <i>mkg./cm.³</i>
1	2534	4780	24.2	52	10.37
	2119				
6	2706	4931	24.0	50	10.48
	2425				
15	2706	4775	27.4	50	11.95
	2483				
20	2801	5093	26.4	48	12.14
	2559				

Table X.

Impact Test—Series G.

June 2 and 3, 1915.

Room Temperature 21.2° to 24.6°C.

No. of Test Piece	Dia. of Hole <i>mm.</i>	Total Work in Fracture <i>mkg.</i>	Specific Work in Fracture <i>mkg./cm.²</i>	Do. mean <i>mkg./cm.²</i>
3	3.60	32.36	7.19	4.62
5		9.18	2.04	
12	3.80	9.18	2.04	2.10
13		9.72	2.16	

No. of Test Piece	Dia. of Hole mm.	Total Work in Fracture mkg.	Specific Work in Fracture mkg./cm. ²	Do. mean mkg./cm. ²
8	3.90	10.26	2.28	2.21
9		9.59	2.13	
2	4.00	9.45	2.10	2.13
17		9.72	2.16	
7	4.10	9.45	2.10	1.95
10		8.10	1.80	
11	4.20	9.05	2.01	1.97
14		8.64	1.92	
16	4.40	10.79	2.40	5.63
18		39.85	8.86	
4	4.60	8.91	1.98	2.58
19		14.33	3.18	

Table XI.

Statical Test—Series H.

June 11 and July 22, 1915.

Room Temperature 21.7° to 31.7°C.

No. of Test Piece	Yielding Point kg./cm. ²	Tensile Strength kg./cm. ²	Breaking Extension %	Area Contraction %	Work in Fracture mkg./cm. ³
1	2708	4619	25.7	49.8	6.98
6	2226 1996	4305	27.9	46.8	7.23
15	3026 2485	4459	26.0	52.2	6.75
20	2709 2470	4459	26.4	55.3	6.97

Table XII.

Impact Test—Series H.

In Summer of 1915.

Room Temperature 33.1°C.

No. of Test Piece	Dia. of Hole mm.	Total Work in Fracture mkg.	Specific Work in Fracture mkg./cm. ²	Do. mean mkg./cm. ²
3	3.60	38.46	8.55	5.23
5		8.57	1.90	
12	3.80	10.37	2.30	2.15
13		9.02	2.00	
8	3.90	11.26	2.50	2.60
9		12.15	2.70	
2	4.00	15.52	3.45	5.32
17		32.32	7.18	
7	4.10	10.07	2.24	5.33
10		37.85	8.42	
11	4.20	10.37	2.30	4.56
14		30.62	6.81	
16	4.40	44.10	9.80	9.09
18		37.61	8.37	
4	4.60	10.37	2.30	5.26
19		36.99	8.22	

III. Influence of Straining.

By straining materials at ordinary temperature, the toughness decreases in general, but the quality is improved by heating them at a certain temperature.

Since this fact was reported elsewhere,¹⁾ the author has paid continual attention to the interesting behavior of strained materials. Goerens²⁾ work may be mentioned as a valuable contribution throwing light on the general features of the problem, but impact resistance seems to have escaped his categories. The following tests may serve to fill this gap.

The specimens were taken from low carbon steel bars in market as used for parts subject to cold strain, and they were annealed at a certain temperature as noted below. The impact test was performed on the material in the following three conditions:

- a. as annealed,
- b. as annealed and strained,
- c. as annealed, strained, and again annealed.

To strain the specimens to be tested in the conditions *b* and *c*, a testing machine was used, and they were pulled beyond the yielding point until a certain amount of strain was reached.

Series M.

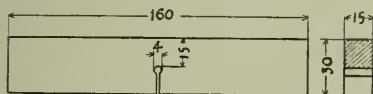
The original bar from which the test pieces were prepared, was a rolled bar of rectangular section, 38 mm. $\times$ 16 mm., and the

1) Memoirs of the College of Science and Engineering, Kyoto Imperial University, Vol. V, No. 9, 1913, p. 369.

2) The Iron and Steel Institute, Carnegie Scholarship Memoirs, Vol. III, 1911, p. 320.

carbon content was 0.048%. The first annealing took place at 900°C. for half an hour, while the temperatures of the second annealing were of two grades, viz. 920° and 770°C. The two extreme portions of the bar were used for preparing flat extension test pieces, Nos. 1 and 14, on which the statical test was performed as recorded in Table XIII. From the remaining material forming the greater part of the original bar 12 impact test pieces were made, and they were numbered from 2 to 13 in their order in the original bar. The impact test pieces, Fig. 14,

Fig. 14
M, N.



were notched bars of which the diameter of holes was 4 mm. in all cases, and the cross sectional area at the narrowest part was 15 mm. ×

15 mm. The results of the impact and hardness tests are collected in Table XIV, which shows the loss of toughness by straining, and its recovery by the second annealing. Further it appears that 770°C. was high enough for the recovery.

Series N.

The preparation of the specimens, Fig. 14, was similar as Series *M*, except that the strain was varied from 1 to 9%, while the second annealing took place at 920°C. for all the pieces. Moreover, the carbon content was found to be 0.037%. The results of the statical and impact tests are given in Tables XV and XVI, respectively. The latter table includes also the results of the hardness tests.

Fig. 15 representing the hardness and the specific work in

fracture as functions of strain, shows an interesting property accompanied by the coarse structure of strained and annealed metals, to which attention has been paid by several authors.¹⁾

The natural size photograph, Fig. 16, showing the fractured surfaces and the microphotographs, Figs. 17 to 20, of the longitudinal sections may serve for explaining the foregoing results. Here the following remarks may be added.

Fig. 17, test piece *N* 2, condition *a*.

Fig. 18, „ „ „ *N* 9, condition *b* (% strain=9).

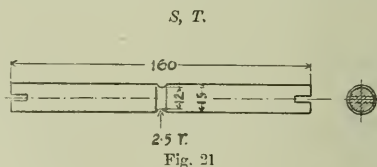
Fig. 19, „ „ „ *N* 10, condition *c* (% strain=9),
grains partly coarsened.

Fig. 20, same piece as Fig. 19, showing the part, where the grains are much coarsened.

Charpy²⁾ points out that the grain growth takes place in the temperature range 650° to 800°C. The fact that the coarsening occurred at higher temperature in the present experiment, might probably be due to the nature and the treatment of the material.

Series S.

The material was a rivet iron bar containing 0.063% carbon, and it was tested in the three conditions *a*, *b*, and *c* as stated before with Prof. Matsumura's repeated impact testing machine. The particulars of the test piece, Fig. 21, were as follows:



1) Charpy, Sauveur, etc. Cp. Engineering, Sept. 20, 1918.

2) Rev. de Metallurgie, 1910.

110^{cm}

Fig. 15. b. Annealed and strained.
 c. Annealed, strained,
 and annealed.

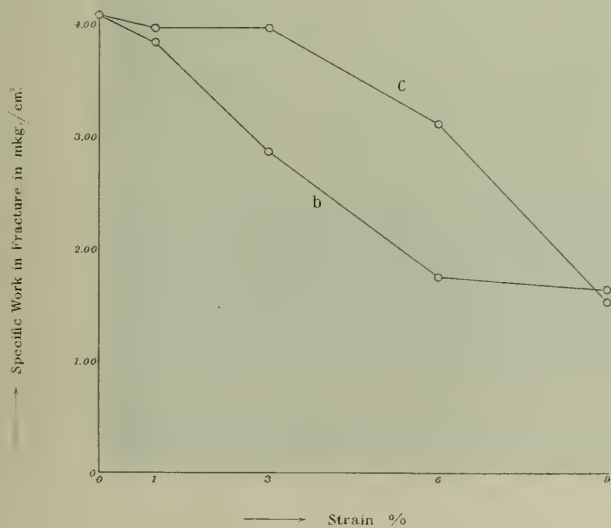
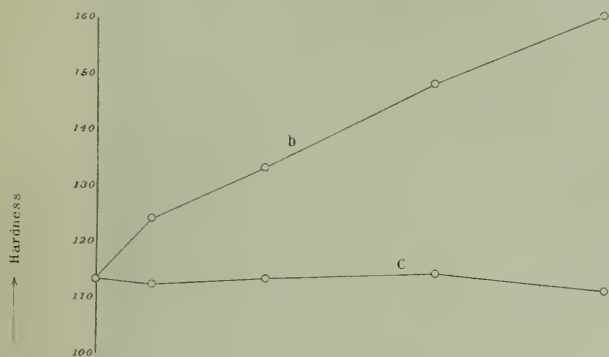


Fig. 16. Series N.

11^D

Annealed.

Annealed
& strained.

Annealed,
strained,
& annealed.

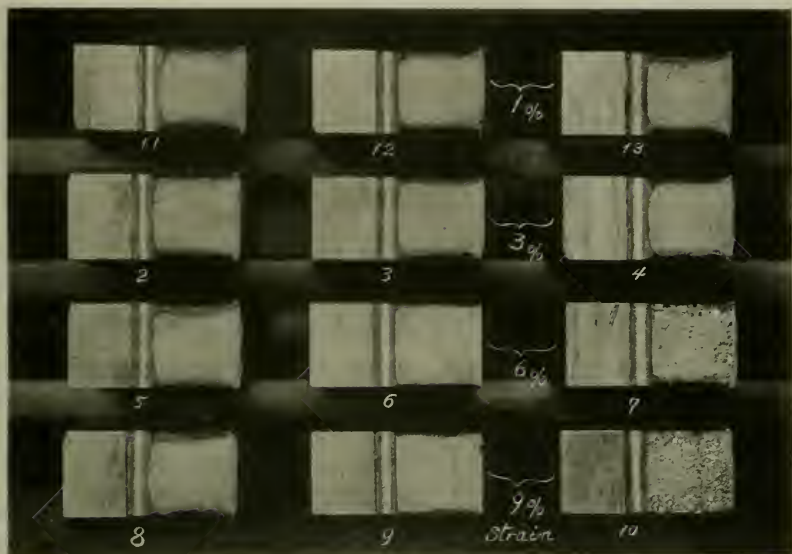


Fig. 22. T 12.

About 1.85× Natural Size.





Fig. 17. N 2. $\times 75$



Fig. 18. N 9. $\times 75$

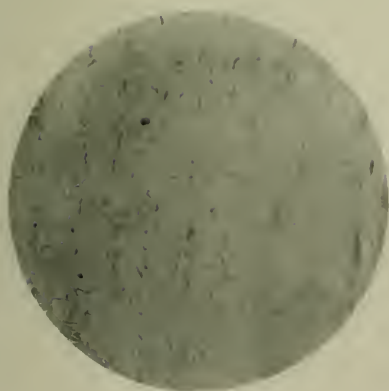


Fig. 19. N 10. $\times 75$



Fig. 20. N 10. $\times 75$

Length	...	...	...	...	...	160	<i>mm.</i>
Diameter	...	...	...	...	...	15	<i>mm.</i>
Do. at the bottom of groove...	...	...	...	...	...	12	<i>mm.</i>
Radius of groove	...	...	...	...	...	2.5	<i>mm.</i>

The energy of each blow of hammer was adjusted at 50 *cmkg.* in the present series.

The first and the second annealing took place at 920°C. for one hour, and the strain was varied from 1 to 11%. But the relation between the amount of strain and the number of blows endured by the material was not clear, though great strain appeared vaguely to correspond to a small number of blows in the condition *b* as a whole. Thus the mean values of the number of blows were calculated for each condition, disregarding the difference in the amount of strain. The final results are shown in Table XVII.

Series T.

The material was the same as in Series *S*, the percentage of carbon content being 0.065. The first annealing took place at 920°C. for one hour, and the strain was 8% in all pieces, while the second annealing was made at various temperatures, viz. 450, 500, 600, 700, 800, 900, and 950°C.

The energy of each blow was made equal to 30 *cmkg.*, and the average number of blows endured by the material in the conditions *a* and *b* were simply calculated as shown in Table XVIII. As to the condition *c* no definite law could be found for the annealing temperature under 600°C. So the mean value obtained from the three tests is given in the table. But with higher annealing temperature, the number of blows became con-

siderably smaller than the value just referred to until the temperature reached 900°C . This result is explained again by the grain coarsening of strained material by annealing. Lastly the material annealed at 900°C . was as tough as the original one, but that at 950°C . was rather fragile. The cause of the latter phenomenon is not evident, but the former one shows the recovery of the material from the strained condition by heating it above A_3 point.

The photographs, Figs. 22 to 26, may be referred to by comparing the following remarks.

Fig. 22, enlarged view (about $1.85\times$ natural size) of test piece T 12, condition c , heated to 700°C . after straining, showing the numerous mat spots observed on the turned surface before the test. Similar phenomenon occurred in T 15 mentioned below.

Fig. 23, test piece T 12 above stated. Grains coarsened.

Fig. 24, test piece T 15, condition c , heated to 800°C . after straining. Grains coarsened.

Fig. 25, test piece T 18, condition c , heated to 900°C . after straining. Grains fine.

Fig. 26, another part of the same test piece as Fig. 25, reheated to 950°C . Grains fine.

It is to be observed that in the present material 900°C . was too high for the grain growth.

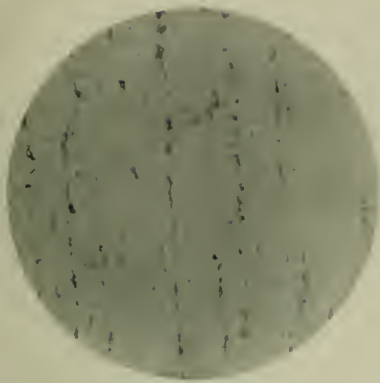


Fig. 23. *T* 12. $\times 75$

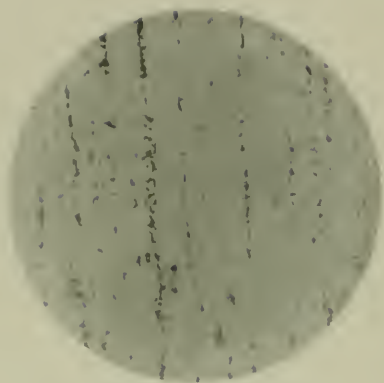


Fig. 24. *T* 15. $\times 75$

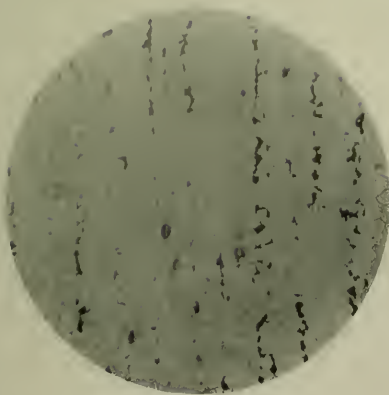


Fig. 25. *T* 18. $\times 75$

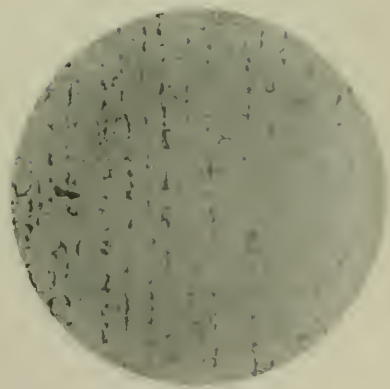


Fig. 26. *T* 15'. $\times 75$

Table XIII.
Statical Test—Series M.

June 2, 1916.

Room Temperature 24° to 25°C.

No. of Test Piece	Yielding Point <i>kg./cm.²</i>	Tensile Strength <i>kg./cm.²</i>	Breaking Extension %	Area Contraction %	Work in Fracture <i>mkg./cm.³</i>
1	3050 2700	4320	29.4	66	9.8
14	2910 2690	4380	27.5	65	9.0

Table XIV.
Impact and Hardness Tests—Series M.

1916.

Room Temperature during the Impact Test 28°C.

No. of Test Piece	Sp. Work in Fracture <i>mkg./cm.²</i>			Strain %	Brinell's Hardness	Remarks
	Annealed	Annealed and Strained	Annealed, Strained, and Annealed			
2	> 9.15			—	122	not broken
3		1.13		2.5	141	
4			13.82	2.5	123	
5	6.09			—	120	
6		0.97		2.5	141	Do. at 770°C. not broken
7			> 14.80	2.4	124	
8	2.30			—	120	
9		1.33		2.6	142	
10			2.00	2.7	123	2nd Annealing at 920°C.
11	15.51			—	122	
12		1.23		2.2	131	
13			15.98	2.6	123	
mean	> 8.26	1.17	> 11.65	2.5	121 139 123	Do. at 770°C.

TABLE XV.

Statical Test—Series N.

June 2 and 3, 1916.

Room Temperature 23° to 25°C.

No. of Test Piece	Yielding Point <i>kg./cm.²</i>	Tensile Strength <i>kg./cm.²</i>	Breaking Extension %	Area Contraction %	Work in Fracture <i>mkg./cm.³</i>
1	2890 2620	4210	30.5	65	9.9
14	2840 2610	4180	29.5	63	9.3

TABLE XVI.

*Impact and Hardness Tests—Series N.*1916. ^a

Room Temperature during the Impact Test 28°C.

No. of Test Piece	Sp. Work in Fracture <i>mkg./cm.²</i>			Strain %	Brinell's Hardness	Remarks
	Annealed	Annealed and Strained	Annealed, Strained, and Annealed			
11	3.96			—	112	←Fracture coarsely granular
12		3.84		1	124	
13			3.96	1	112	
2	4.08			—	114	
3		2.87		3	133	
4			3.96	3	113	
5	4.32			—	114	
6		1.76		6	148	
7			3.11	6	114	
8	3.96			—	113	
9		1.64		9	160	
10			1.51	9	111	
mean	4.08	—	—	—	113 — —	

Table XVII.

Repeated Impact Tests—Series S.

Dec. 1916.

Room Temperature = 15° to $18^{\circ}C$.

Energy of Each Blow = 50 cmkg.

State of Material	No. of Blows	Remarks
Annealed	435	mean of 6 tests*
Annealed & Strained (% Strain = 1 to 11)	389	mean of 7 tests (Strains all different)
Annealed, Strained, & Annealed	396	mean of 7 tests

* A test piece was rejected on account of great deformation after 81 blows.

Table XVIII.

Repeated Impact Tests—Series T.

Dec. 1916.

Room Temperature = 12.5° to $19.0^{\circ}C$.

Energy of Each Blow = 30 cmkg.

State of Material	No. of Blows	Remarks
Annealed	1010	mean of 5 tests*
Annealed & Strained (% strain = 8)	940	mean of 7 tests
Annealed, Strained, & Annealed at about 450° , 500° , & $600^{\circ}C$.	1013	mean of 3 tests (Temps. all different)
Do. at $700^{\circ}C$.	496	single test
Do. at $800^{\circ}C$.	553	"
Do. at $900^{\circ}C$.	1104	"
Do. at $950^{\circ}C$.	572	"

* A test piece was bent during turning and rejected. Another piece endured very small number of blows (414). The cause is not clear, and it was also rejected.

Summary.

1. The variation of the work in fracture with the diameter of notch holes in mild steel impact test pieces for testing machines of pendulum type has been made the subject of experiment with the result that the error ± 0.1 mm. made in drilling a hole 4 mm. in diameter may give rise to a deviation of some $\pm 1.5\%$ in the final result. This may be considered as quite insignificant in impact testing.

2. Some results of impact tests on mild steel specimens with banded structure are reported to show that the banding is not probably the source of weakness, but that the local concentration of less plastic material may give rise to the irregular behavior in notched bar impact tests.

3. The effect of cold straining on the toughness of ductile iron was investigated, and it was found that the toughness impaired by cold strain was restored by heating. The temperature at which the recovery first took place was not accurately determined, but $770^{\circ}\text{C}.$ was found to be high enough in one case (Series *M*), and 600° was sufficient in the other case (Series *T*).

4. Although the result of repeated impact tests can not be directly compared with that of the pendulum impact tests, the loss of toughness by cold straining was found to be not very considerable in the material experimented on in the former way, Series *S* and *T*.

5. The fact that the grain growth takes place in strained material as the result of heating is not new, but the present notes can not be concluded without referring to the said interesting phenomenon, which occurred naturally in the course of the experiment.

Fukuoka, July, 1919.

ERRATUM.

Page 130, line 7, for 'b' read ' b_s '.

Prof. Akimasa Ono, *Kogakuhakushi*.

(Received July 7. 1920.)

Previous Works.

Experimental Investigation.

The investigation on the fatigue of materials under one kind of stress varying between certain limits has been performed by various authors. The famous experiments of Wöhler²⁾ and Bauschinger^{3,4)} are now the classical works, and the results are recorded in many literatures. Of later authors so far as the present author is aware, the works of Reynolds and Smith⁵⁾; Stanton and Bairstow⁶⁾; Smith⁷⁾; Bairstow⁸⁾; Eden, Rose and

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2) Wöhler's Experiments on the Fatigue of Metals. Engineering, Vol. XI, 1871.

3) J. Bauschinger: Ueber die Veränderung der Elasticitätsgrenze und der Festigkeit des Eisens und Stahls durch Strecken und Quetschen, durch Erwärmen und Abkühlen und durch oftmal wiederholte Beanspruchung. Mitteilungen aus dem mechanisch-technischen Laboratorium der K. technischen Hochschule in München, XIII. Heft, 1886.

4) A. Föppl: Dauerversuche von Bauschinger, ausgeführt in den Jahren 1886-1893, Mitteilungen, München, XXV. Heft, 1897.

5) O. Reynolds and J. H. Smith: On a Throw-Testing Machine for Reversals of Means Stress. Phil. Trans. Roy. Soc., A, Vol. 199, 1902.

6) T. E. Stanton and L. Bairstow: On the Resistance of Iron and Steel to Reversals of Direct Stress. Proc. Inst. Civil Engineers, Vol. CLXVI, 1906.

7) J. H. Smith: Some Experiments on Fatigue of Metals. Jour. Iron and Steel Institute, 1910, No. II.

8) L. Bairstow: The Elastic Limits of Iron and Steel under Cyclical Variations of Stress. Phil. Trans. Roy. Soc., A, Vol. 210, 1911.

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Fukuoka, July, 1919.

Fatigue of Steel under Combined Bending and Torsion.¹⁾

By

Prof. Akimasa Ono, *Kogakuhakushi.*

(Received July 7, 1920.)

Previous Works.

Experimental Investigation.

The investigation on the fatigue of materials under one kind of stress varying between certain limits has been performed by various authors. The famous experiments of Wöhler²⁾ and Bauschinger^{3,4)} are now the classical works, and the results are recorded in many literatures. Of later authors so far as the present author is aware, the works of Reynolds and Smith⁵⁾; Stanton and Bairstow⁶⁾; Smith⁷⁾; Bairstow⁸⁾; Eden, Rose and

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Cunningham¹⁾; Turner²⁾; Mason³⁾; and others may be mentioned.

The behaviour of materials under varying combined stresses has been little experimented on, though we can mention the work of Stanton and Batson^{4,5)} of the National Physical Laboratory. In their experiments mild steel specimens were tested in a fatigue testing machine specially designed for experimenting materials under combined bending and torsion varying between two limits of equal magnitudes and of opposite directions, or under either kind of loading acting separately. The stress range being determined for certain ratio of the twisting and bending moments, the criterion for the failure may be represented by the maximum shearing stress in one series, but this conclusion was not supported in further series of investigation.

Mathematical Investigation.

The extension of the calculation relating to the strength of materials, to the case of varying stresses, was performed by Paul Roth.⁶⁾ He assumed the allowable stress to obey the law usually taken for granted in the design of machine elements (cp. Bach: Die Maschinen—Elemente). Thus let us denote by

1) E. M. Eden, W. N. Rose, and F. L. Cunningham: The Endurance of Metals. Proc. Inst. Mech. Engineers, 1911.

2) L. B. Turner: The Strength of Steels in Compound Stress, and Endurance under Repetition of Stress. Engineering. Vol. XCII, 1911.

3) W. Mason: Alternating Stress Experiments, Engineering, Vol. CIII, 1917.

4) T. E. Stanton and R. G. Batson: On the Fatigue Resistance of Mild Steel under Various Conditions of Stress Distribution. Engineering, Sept. 15, 1916.

5) The National Physical Laboratory in 1916-1917. Engineering, June 22, 1917.

6) Paul Roth: Die Festigkeitstheorien und die von ihnen abhängigen Formeln des Maschinenbaues. Zeitschrift für Mathematik und Physik, 1902.

$f_{\max}, f_{\min}$	the upper and lower limits of stresses, assuming that the magnitude of $f_{\min}$ is not greater than that of $f_{\max}$,
$\gamma = \frac{f_{\min}}{f_{\max}}$	the ratio of the limits of stresses,
k	the allowable stress in the case when the stress varies in the above limits,
k_c	a constant of the nature as described below.

Then

$$k = k_c \left(3 - \frac{f_{\max} - f_{\min}}{f_{\max}} \right),$$

or it is the same thing to write

$$k = k_c (2 + \gamma), \quad (1)$$

which gives

- (case a) $k = 3 k_c$, when $\gamma = +1$, i.e. the stress is steady,
 (case b) $k = 2 k_c$, when $\gamma = 0$, i.e. the stress is varying between zero and a certain magnitude.
 (case c) $k = k_c$, when $\gamma = -1$, i.e. the stress is varying between positive and negative limits of equal magnitudes.

The equation (1) is essentially the same equation taken by Roth. The author calculated the value of the ratio $\frac{f_{\max} - f_{\min}}{f_{\max}}$ in various directions for rotating shafts of circular section under the bending moment M_b and the steady twisting moment M_t , and substituted it in the equation (1). Then the minimum diameter could be determined for each direction, the greatest of all being the required diameter. If the material obeys the strain hypothesis, then it appears that so long as $M_t < 3.2 M_b$, we obtain the greatest diameter, when the extension in the direction of the axis is taken as the basis, i.e. the twisting moment may be

neglected. Again, if the material is of such nature that the shearing stress hypothesis is true, then the stress acting on an elementary area inclined at 45° to the axis shall be taken so long as $M_t < 2.4 M_b$, thus the twisting moment being of no effect. The author proceeds to treat the case, where not only the bending moment but also the torsional moment undergoes the periodic change by assuming the shearing stress hypothesis.

A similar subject was treated by R. W. Bailey¹⁾ by assuming the shearing stress hypothesis. He took a shaft subject to a completely reversing bending moment and a varying twisting moment, and calculated the statical shearing stress of equal destructiveness by the following equation

$$f = \frac{f_a}{8}(4 + 3\gamma + \gamma^2), \quad (2)$$

where

f_a = statical shearing stress,

f = limiting fatigue stress,

$$\gamma = \frac{\text{min. shearing stress}}{\text{max. shearing stress}}.$$

This equation gives

$$f = f_a \quad \text{for} \quad \gamma = 1 \quad (\text{case a}),$$

$$f = \frac{f_a}{2} \quad \text{for} \quad \gamma = 0 \quad (\text{case b}),$$

$$f = \frac{f_a}{4} \quad \text{for} \quad \gamma = -1 \quad (\text{case c}).$$

The plane at which the above equivalent statical shearing stress becomes a maximum being taken as the plane of failure, he found in the case of the steady twisting moment that for $M_t < 3M_b$ the bending moment alone is responsible for the failure.

1) R. W. Bailey: Ductile Materials under Variable Shear Stress. Engineering, July 27, 1917.

Fatigue Resistance vs. Carbon Content of Steel.

Among the various results of the previous tests we observe certain relation between fatigue resistance and carbon content of steel. Let us first refer to the results of experiments performed by Stanton and Bairstow and by Smith. The former authors used a reciprocating mass machine, while the latter took a revolving mass one. The carbon contents being arranged in their order of magnitudes, and divided into grades of 0.1 % step, the mean value of each step was found to be as shown in the table. Again the fatigue resistance shown below may be considered as the upper limit of a stress range, of which the lower limit is the negative stress of equal magnitude to the upper one. The said stress range may be repeated as often as some 10^6 alterations without fracturing the material.

% C	Fatigue Resistance for Tension kg/cm ²
0.05	1490
0.16	1990
0.27	2110
0.35	2200
0.45	2280
0.57	2670
0.63	2470
0.79	2360

If we represent these experimental results in a diagram with the percentage carbon as abscissa and the stress as the ordinate, the points thus plotted appear to lie rather nearly along two straight lines, *AB* and *BD*, Fig. 1, except the two points corresponding to 0.57 and 0.79 % *C*, these points representing the results obtained from single kind of specimens, respectively. In

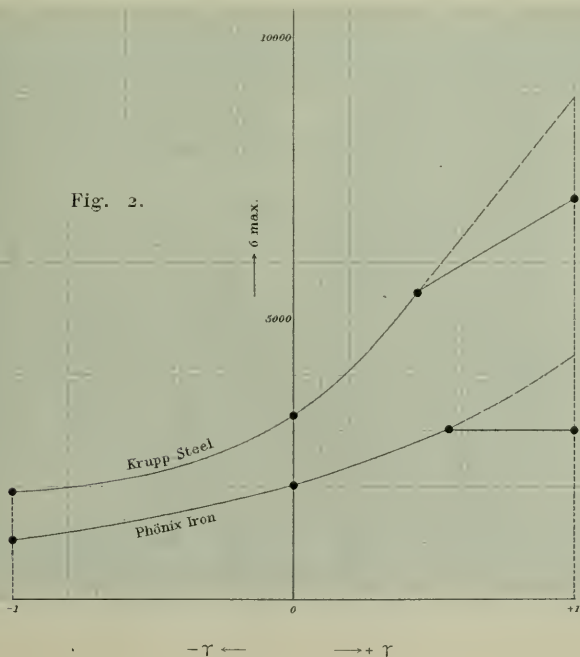
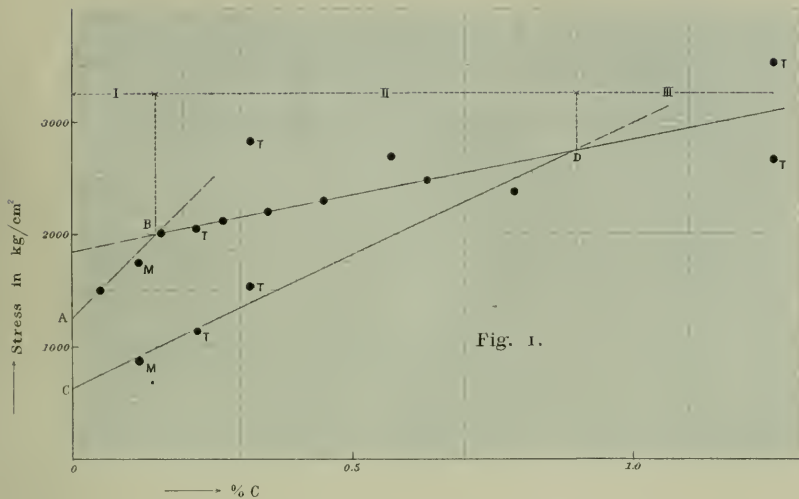
the above diagram we can further plot the results due to Mason and Turner, who experimented the fatigue resistance of steel not only under bending but also under torsion, as summarized below.

% C	Fatigue Resistance kg/cm^2		Ratio $\frac{\text{Torsion}}{\text{Bending}}$	Author
	Bending	Torsion		
0.12	1733	887	0.51	Mason
0.15~0.30	2040	1125	0.55	Turner
0.32	2810	1547	0.55	„
1.25	3520	2670	0.76	„

The points due to these authors are designated by their initials in the same figure.

These experiments were performed with different testing machines, and the resistances above given were determined not in a similar manner to the foregoing experiments. So conformity can not be expected between the former and latter experiments. Thus the two series of Turner's tests show rather considerable deviation from the straight line above drawn. But it appears that the ratio of the torsional stress and bending stress is nearly 0.5 for very low carbon steel and increases with the carbon content. Thus if we conventionally draw a straight line of which the ordinate is half that of AB and produce it, then the straight line CD will intersect with BD at D of which the abscissa is equal to about 0.9 % C.

The diagram thus constructed may be used for interpreting the manner of fracture of steel, for the time being, until more complete ones can be drawn. In part I of the diagram the tensile stress equal to twice the shearing stress is equally as dangerous as the latter, i.e., the steel with % C zero to 0.15 or thereabout fails probably by shear under both kinds of stresses. The supposition that the material in this region fails by separa-



tion under bending and by shear under torsion may appear to be correct at first sight, but the sharp bend of the lines *ABD* at *B* seems to suggest that this is not presumably right. Next in part II the steel fails under the tensile stress less than twice the shearing stress but greater than the latter with equal degree of danger. So the material fails by separation under tensile stress before sliding occurs. This is the case in hypoeutectoid steel of not very small carbon content. Lastly for higher carbon steel as represented by III, the failure takes place by separation even in the case of torsion, as the material is weaker under normal stress than shearing stress.

According to the foregoing interpretation of the diagram, the criterion for the failure of steel belonging to part I appears to be shearing stress, while that for part III is probably the tensile stress. Thus there remains a wide field as represented by II in which neither the shearing stress nor the tensile stress alone is sufficient for determining the failure.

Here it should be added that the fracture of specimens can hardly give reliable information as to the hypothesis of failure. In soft material the fracture appears to be in accord with the shear hypothesis as explained by Turner with regard to tube steel of 0.15 to 0.30 % C, but in solid specimens, subject to repeated bending, the fracture takes place rather along a cross section and not along an inclined plane.

It should be borne in mind that break down may take place either by the sliding or by the separation of the two parts mutually in contact with each other to form one body, but the fatigue is set in before the break down occurs. According to the microscopical investigation of Stanton and Bairstow¹⁾ on carbon

1) loc. cit.

steel containing up to 0.64 % C, the fatigue is due to the slipping appearing chiefly in the region of ferrite. Thus the break down is the result either of sliding or of separation according to the condition of one being more easily produced than the other in the material in which many slip lines are generated by the repeated action of stress, so far as the foregoing experiments suggest. Accordingly, the fatigue and break down are the two successive phenomena to be observed, and the easiness of introducing these facts determines the fatigue resistance. So the cross sectional fracture of bending specimens does not necessarily mean that the material follows the principal stress hypothesis.

Formula for Fatigue Resistance.

The upper stress which can be repeated a certain number of times, depends as is well known, upon the stress range, and therefore it may be considered to be determined by the lower stress or by the ratio of the two stresses. Thus denoting the stress in general by f ,

$$f_{\max} = \varphi (f_{\min}) \quad (3)$$

or

$$f_{\max} = \psi (\gamma), \quad \text{where } \gamma = \frac{f_{\min}}{f_{\max}}. \quad (4)$$

Any convenient formula either of the form (3) or (4) may be used to express the variation of $f_{\max}$ provided that it answers the purpose of representing experimental results. For example, the equations (1) and (2) already introduced belong to type (4), though (1) stands for the allowable stress instead of the maximum stress as here understood, while the simplest form of (3) is the linear equation with respect to $f_{\min}$ as mentioned below.

It is to be remarked that the quadratic equation

$$f_{\max} = a + b\gamma + c\gamma^2, \quad (5)$$

of which (2) is a particular example, appears to be roughly consistent with experimental results, if the constants are properly chosen. But this equation is still incapable of representing the relation as denoted with full lines in Fig. 2, which represents the classical experiments of Wöhler, as the curves show points of inflexion. The sudden bend in the part $\gamma > 0$ is due to the reason that the stress corresponding to $\gamma = 1$ is taken as equal to the technical stress, which is much smaller than the actual stress, the former being calculated with respect to the original area of the piece in spite of the considerable area contraction before fracture. Thus if we take the stress referred to the contracted area, the curves will run as shown with broken lines.

Thus a formula suitable to represent both the full and broken lines appears to be desirable, although it depends upon the circumstances whether we have to choose one or the other of the curves. For this purpose the following equation may be mentioned.

$$f_{\max} = \frac{b}{1 - a_1\gamma + a_2\gamma^2}. \quad (6)$$

This seems in the author's view better fitted to the case than (5). Because, if we differentiate $f_{\max}$ with respect to γ ,

$$\frac{d^2 f_{\max}}{d\gamma^2} = 0, \text{ when}$$

$$\gamma = \frac{a_1}{2a_2} \left[1 \pm \sqrt{\frac{1}{3} \left(\frac{4a_2}{a_1^2} - 1 \right)} \right], \quad (7)$$

which is real when $4a_2 \geq a_1^2$.

Particularly consider the case when the three typical $f_{\max}$'s

are in the ratios 1 : 2 : 3 and 1 : 2 : 4 corresponding approximately to the full and broken lines in Fig. 2, respectively. Then the values of a_1 and a_2 will be as follows :

$$\text{ratios } 1 : 2 : 3, \quad a_1 = \frac{2}{3}, \quad a_2 = \frac{1}{3}.$$

$$,, \quad 1 : 2 : 4, \quad a_1 = \frac{3}{4}, \quad a_2 = \frac{1}{4}.$$

The curves drawn with full lines in Fig. 3 show the ratio

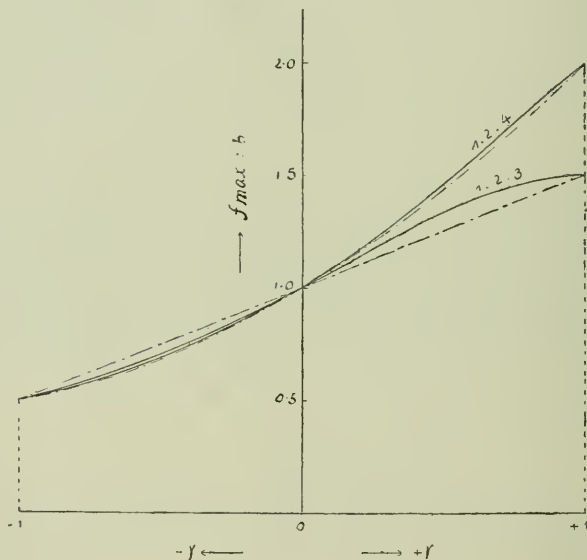


Fig. 3.

of $f_{\max} : b$ for the above two cases, while the chain lines refer to the following equations of the type (5).

$$\text{ratios } 1 : 2 : 3, \quad f_{\max} = b(1 + \frac{1}{2}\gamma),$$

$$,, \quad 1 : 2 : 4, \quad f_{\max} = b(1 + \frac{3}{4}\gamma + \frac{1}{4}\gamma^2).$$

Now in the present problem we have to consider the bending

and torsion, which do not accompany considerable area contraction; so it may probably be right to chose a curve as represented with broken lines in Fig. 2. Thus taking the equation

$$r_{\max} = \frac{b}{1 - \frac{3}{4}\gamma + \frac{1}{4}\gamma^2} \quad (8)$$

as the basis of calculation, it is to be observed that there will be no appreciable difference between the present equation and the latter of the above two equations. But the present equation has a further advantage as it reduces to the form (3) when $\alpha_2=0$, because in that case

$$\begin{aligned} r_{\max} &= \frac{b}{1 - \alpha_1\gamma} = \frac{b}{1 - \alpha_1 \frac{f_{\min}}{f_{\max}}} \\ \text{or} \quad f_{\max} &= b + \alpha_1 f_{\min}. \end{aligned} \quad (9)$$

This is the linear equation, which may be conveniently used for the sake of simplicity as suggested by Goodman, although the conformity with experimental results can not be very satisfactory.

Alternating Combined Stress.

Let us consider a rod subject to varying stresses due to bending and torsion. First we assume that the number of repetitions of stresses until fracture is solely dependent on the equivalent shearing stress b_s as determined by the equation (6), viz.

$$b_s = \tau_{\max} (1 - \alpha_1\gamma + \alpha_2\gamma^2),$$

γ standing for the ratio $\frac{\tau_{\min}}{\tau_{\max}}$.

This b_s is equal to $\tau_{\max}$ when $\gamma=0$.

Now the shearing stress induced in the plane inclined at an angle φ to the cross section of the rod is generally

$$\tau_{\varphi} = \frac{\sigma}{2} \sin 2\varphi - \tau \cos 2\varphi,$$

where σ and τ are the normal and shearing stresses acting at a point of the cross section. Assume that the stresses vary in such a way that σ varies between $\pm\sigma$ and $\pm x\sigma$, while τ varies between τ and $y\tau$, x and y being numerical coefficients lying between 1 and -1 . Here the double signs before σ signifies the positive and negative stresses at the extremities of the cross section. As the sense of τ_{φ} is immaterial, let us assume that for a definite angle φ less than 45° , the maximum and minimum stresses will be

$$\tau_{\max} = \frac{\sigma}{2} \sin 2\varphi + \tau \cos 2\varphi$$

and

$$\tau_{\min} = -\frac{x\sigma}{2} \sin 2\varphi + y\tau \cos 2\varphi,$$

respectively.

Accordingly, putting

$$\gamma = \frac{x \frac{\sigma}{2} \sin 2\varphi + y \tau \cos 2\varphi}{\frac{\sigma}{2} \sin 2\varphi + \tau \cos 2\varphi},$$

we have

$$\begin{aligned} h_s &= \left(\frac{\sigma}{2} \sin 2\varphi + \tau \cos 2\varphi \right) (1 - a_1 \gamma + a_2 \gamma^2) \\ &= (1 - a_1 x) \frac{\sigma}{2} \sin 2\varphi + (1 - a_1 y) \tau \cos 2\varphi \\ &\quad + \frac{a_2 \left(x \frac{\sigma}{2} \sin 2\varphi + y \tau \cos 2\varphi \right)^2}{\frac{\sigma}{2} \sin 2\varphi + \tau \cos 2\varphi}. \end{aligned} \tag{10}$$

Now, if the repeated shearing stress is great enough, the

failure of material will according to the present hypothesis take place in such a plane for which the value of b_s is greatest, because such a value of b will induce the fracture more quickly than in any other planes. A maximum or a minimum value of b_s occurs, when

$$A \tan^3 2\varphi + B \tan^2 2\varphi + C \tan 2\varphi + D = 0, \quad (11)$$

where

$$\begin{aligned} A &= [1 - a_1 y - a_2 (x^2 - 2xy)] \frac{\tau}{\sigma}, \\ B &= -\frac{1}{2}(1 - a_1 x + a_2 x^2) + (1 - a_1 y + a_2 y^2) \left(\frac{2\tau}{\sigma} \right)^2, \\ C &= -(1 - a_1 x + a_2 x^2) \frac{2\tau}{\sigma} + \frac{1}{2}(1 - a_1 y + a_2 y^2) \left(\frac{2\tau}{\sigma} \right)^3, \\ D &= -2[1 - a_1 x + a_2 (2xy - y^2)] \left(\frac{\tau}{\sigma} \right)^2. \end{aligned}$$

The above calculation may clear up the variation of b_s when $\varphi < 45^\circ$. When $\varphi > 45^\circ$, the second term on the right side of τ_φ is positive; so τ_φ is numerically greatest for the greatest positive σ , and for the greatest positive τ . Thus

$$\tau_{\max} = \frac{\sigma}{2} \sin 2\varphi - \tau \cos 2\varphi,$$

while

$$\tau_{\min} = -\frac{x\sigma}{2} \sin 2\varphi - y\tau \cos 2\varphi.$$

With

$$r = \frac{x \frac{\sigma}{2} \sin 2\varphi - y\tau \cos 2\varphi}{\frac{\sigma}{2} \sin 2\varphi - \tau \cos 2\varphi},$$

we have

$$\begin{aligned} b_s &= \left(\frac{\sigma}{2} \sin 2\varphi - \tau \cos 2\varphi \right) (1 - a_1 r + a_2 r^2) \\ &= (1 - a_1 x) \frac{\sigma}{2} \sin 2\varphi - (1 - a_1 y) \tau \cos 2\varphi \end{aligned}$$

$$+ \frac{a_2 \left(x \frac{\sigma}{2} \sin 2\varphi - y \tau \cos 2\varphi \right)^2}{\frac{\sigma}{2} \sin 2\varphi - \tau \cos 2\varphi}. \quad (12)$$

The particular case of most common occurrence is such that the normal stress due to bending varies between the positive and negative limits of the equal magnitudes, and therefore $x=-1$ in the above expressions.

By assuming a_1 and a_2 at 0.75 and 0.25, respectively, the variation of b with φ in the case when $x=-1$ and $y=1$ is graphically shown in Fig. 4, and the maximum value of b_s for assigned values of the ratio $\frac{\tau}{\sigma}$ can be readily seen.

For the negative value of φ we can construct a similar diagram, but this is not necessary, because nothing new will be obtained by this process.

According to the present diagram, the curve for $\sigma = \frac{\tau}{1.6}$ marks the boundary of two regions, in one of which the value of b_s is not greater than σ corresponding to $\varphi = 45^\circ$, while in the other region b_s becomes greater. In the present problem the shearing stress due to bending varies between $\frac{\sigma}{2}$ and $-\frac{\sigma}{2}$. This stress range is according to the present assumption equivalent to the shearing stress varying between σ and zero, and therefore $b_s = \sigma$. So it appears that a rod subject to a certain amount of alternating bending stress endures the same number of repetitions irrespectively of the steady torsional stress so long as the latter does not surpass 1.6 times the bending stress. This conclusion is not new as already referred to on p. 119 and 120, though the assumption on which the calculation is based has not been tried; still it is here given in order to make the following treatment complete.

130^a

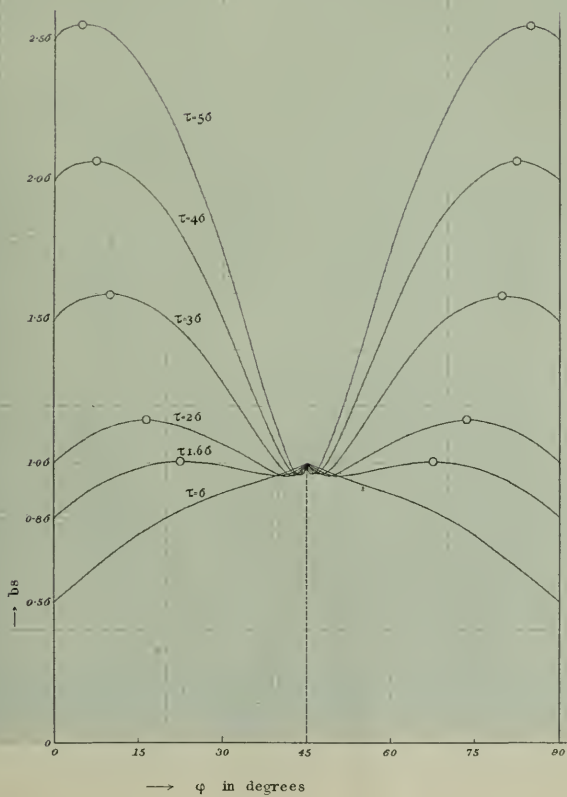


Fig. 4.

Next we assume that the endurance is governed by the equivalent normal stress, which runs as follows :

$$b_n = \sigma_{\max}(1 - a_1 y + a_2 y^2).$$

Now the normal stress at the plane inclined at φ to the cross section of rod is

$$\sigma_{\varphi} = \frac{\sigma}{2}(1 + \cos 2\varphi) + \tau \sin 2\varphi.$$

Thus

$$\sigma_{\max} = \frac{\sigma}{2}(1 + \cos 2\varphi) + \tau \sin 2\varphi,$$

$$\sigma_{\min} = \frac{x\sigma}{2}(1 + \cos 2\varphi) + y\tau \sin 2\varphi,$$

x and y being the numerical constants as before. Accordingly,

if we put $\gamma = \frac{\sigma_{\min}}{\sigma_{\max}}$, then

$$\begin{aligned} b_n &= \left[\frac{\sigma}{2}(1 + \cos 2\varphi) + \tau \sin 2\varphi \right] (1 - a_1 y + a_2 y^2) \\ &= (1 - a_1 x) \frac{\sigma}{2}(1 + \cos 2\varphi) + (1 - a_1 y) \tau \sin 2\varphi \\ &\quad + \frac{a_2 \left[x \frac{\sigma}{2}(1 + \cos 2\varphi) + y \tau \sin 2\varphi \right]^2}{\frac{\sigma}{2}(1 + \cos 2\varphi) + \tau \sin 2\varphi}. \end{aligned} \quad (13)$$

The value of b_n is a maximum or a minimum, when

$$A \tan^4 \varphi + B \tan^3 \varphi + C \tan^2 \varphi + D \tan \varphi + E = 0, \quad (14)$$

where

$$A = (1 - a_1 y + a_2 y^2) \left(\frac{\tau}{\sigma} \right)^3,$$

$$B = [2 - a_1(x + y) + 2a_2 xy] \left(\frac{\tau}{\sigma} \right)^2,$$

$$C = -(1 - a_1 y + a_2 y^2) \left(\frac{\tau}{\sigma} \right)^3$$

$$\begin{aligned}
& + \left[\frac{1-a_1y}{4} + 1-a_1x + \frac{a_2}{2}(xy + \frac{3}{2}x^2) \right] \frac{\tau}{\sigma}, \\
D &= -(1-a_1y+a_2y^2) \left(\frac{\tau}{\sigma} \right)^2 + \frac{1-a_1x+a_2x^2}{4}, \\
E &= -[1-a_1y-a_2x(x-2y)] \frac{\tau}{4\sigma}.
\end{aligned}$$

The variation of b_n with φ for several assigned values of the ratio $\frac{\tau}{\sigma}$ for $x=-1$ and $y=+1$ is graphically represented in Fig. 5, the case of negative φ being left out.

From the above diagram it appears that the maximum value of b_n is a little greater than 2σ in the case when $\frac{\tau}{\sigma} = 2.5$, and that for smaller values of the ratio it remains always less than 2σ . Now the normal stress varying between σ and $-\sigma$ is equivalent to the stress range 2σ and zero according to the present assumption, as $b_n=2\sigma$ in the both cases. So, if a rod is sufficiently strong for alternating bending, then steady torsion may be allowed simultaneously on the same rod without impairing the durability so long as the shearing stress is less than about 2.5 times the normal stress.

This conclusion is qualitatively similar to the foregoing one based on the shearing stress hypothesis. The two hypotheses being considered as the two extreme laws governing the endurance of material, we see that in any kind of material a similar result may be expected. This result wants, however, to be verified by experiments before it is accepted as truth.

Digression on Formulæ for Designing Shafts.

Before entering into the report on the experiments made for the purpose of elucidating the result of the foregoing calculation,

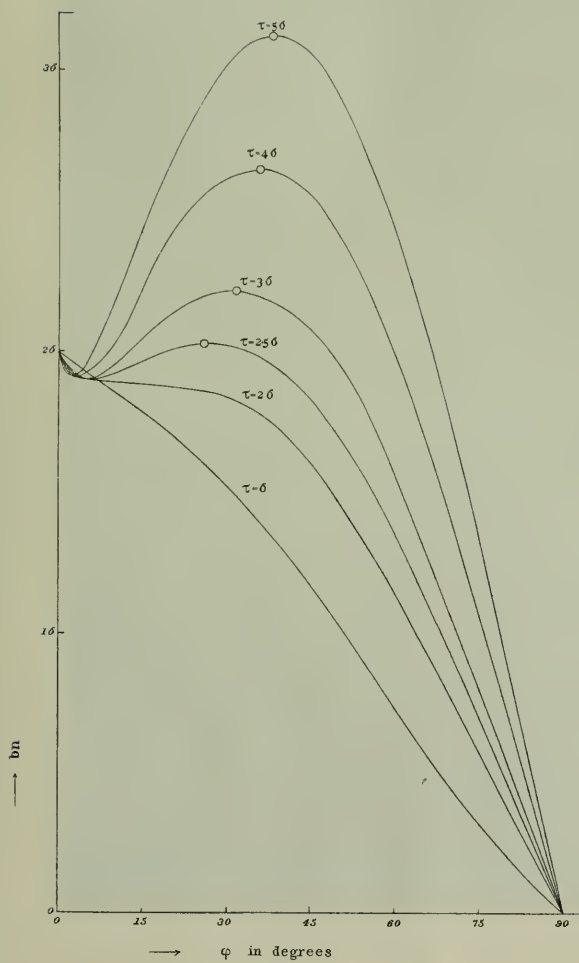


Fig. 5.

let us deduce a formula for the strength of material under varying normal and shearing stresses by assuming the simple relation (9), viz.,

$$f_{\max} = b + a f_{\min},$$

and compare it with that used in designing shafts.

The linear relation between $f_{\max}$ and $f_{\min}$ holds good approximately at least in a certain range. If we, however, assume that this relation is correct for the whole range of the ratio r , then the calculation will be much simplified.

Now putting $r = -1$ in (9),

$$\frac{b}{f_{\max}} = 1 + a$$

or

$$a = \frac{b}{f_{\max}} - 1.$$

For

$$\frac{b}{f_{\max}} = 2, \quad a = 1,$$

and for

$$\frac{b}{f_{\max}} = 1.5, \quad a = 0.5.$$

With the former value of a , the stress ratios for the three typical cases are 1:2:∞; so its use should be limited to the stress variation, of which the lower limit is zero or negative. Again the latter value of a gives the ratios 1:1.5:3 for the three cases.

First, referring to the shearing stress hypothesis,

$$\tau_{\max} = \frac{\sigma}{2} \sin 2\varphi + \tau \cos 2\varphi,$$

and

$$\tau_{\min} = -\frac{\sigma}{2} \sin 2\varphi + y\tau \cos 2\varphi,$$

for $\varphi < 45^\circ$, and $x = -1$.

Thus

$$b_s = (1+a) \frac{\sigma}{2} \sin 2\varphi + (1-ay) \tau \cos 2\varphi.$$

Now

$$\frac{db_s}{d\varphi} = 0,$$

when

$$(1+a) \frac{\sigma}{2} \cos 2\varphi - (1-ay) \tau \sin 2\varphi = 0$$

or

$$\tan 2\varphi = \frac{1+a}{1-ay} \frac{\sigma}{2\tau}.$$

Substituting this in the above expression,

$$\begin{aligned} b_s &= \left[\frac{(1+a)^2 \sigma^2}{(1-ay)^2 4\tau} + (1-ay) \tau \right] \frac{(1-ay) 2\tau}{\sqrt{(1+a)^2 \sigma^2 + (1-ay)^2 4\tau^2}} \\ &= \frac{1}{2} \sqrt{(1+a)^2 \sigma^2 + 4(1-ay)^2 \tau^2}. \end{aligned}$$

But $\frac{b_s}{1+a}$ = limit of stress varying between positive and negative stresses. So putting $k_s = \frac{b_s}{1+a}$, we understand by k_s the limit of stress, which can be repeated a certain number of times without fracture. Then

$$k_s = \frac{1}{2} \sqrt{\sigma^2 + 4 \left(\frac{1-ay}{1+a} \right)^2 \tau^2}. \quad (15)$$

The calculation for the case $\varphi > 45^\circ$ will give rise to the same result.

Again taking the principal stress hypothesis,

$$\sigma_{\max} = \frac{\sigma}{2} (1 + \cos 2\varphi) + \tau \sin 2\varphi,$$

$$\sigma_{\min} = -\frac{\sigma}{2} (1 + \cos 2\varphi) + y\tau \sin 2\varphi,$$

the normal stress being assumed to vary between positive and negative limits of equal magnitudes as before. Thus

$$b_n = (1+a) \frac{\sigma}{2} (1 + \cos 2\varphi) + (1-ay) \tau \sin 2\varphi.$$

Differentiating with respect to φ ,

$$-(1+a) \frac{\sigma}{2} \sin 2\varphi + (1-ay) \tau \cos 2\varphi = 0,$$

or

$$\tan 2\varphi = \frac{1-ay}{1+a} \frac{2\tau}{\sigma}.$$

Substituting this value in the expression for b_n ,

$$\begin{aligned} b_n &= (1+a) \frac{\sigma}{2} \\ &+ \left[(1+a) \frac{\sigma}{2} + (1-ay) \tau \frac{1-ay}{1+a} \frac{2\tau}{\sigma} \right] \frac{(1+a)\sigma}{\sqrt{(1+a)^2\sigma^2 + (1-ay)^2 4\tau^2}} \\ &= (1+a) \left[\frac{\sigma}{2} + \frac{1}{2} \sqrt{\sigma^2 + 4 \left(\frac{1-ay}{1+a} \right)^2 \tau^2} \right]. \end{aligned}$$

But $\frac{b_n}{1+a}$ = limit of stress varying between positive and negative stresses. So putting k_n in the place of $\frac{b_n}{1+a}$,

$$k_n = \frac{\sigma}{2} + \frac{1}{2} \sqrt{\sigma^2 + 4 \left(\frac{1-ay}{1+a} \right)^2 \tau^2}. \quad (16)$$

To establish an equation for a law of failure, which may be considered to lie between the above two extreme cases, it will be necessary to put a postulate combining the normal and shearing stresses beside the law existing in the case of statical stresses.¹⁾ This would bring forth rather arbitrary assumptions, which need experimental support, and such an idea was therefore put aside in the present calculation.

It should be, however, noted that the above formulae are of a similar form to those deduced from the statical calculation, except the coefficient before τ^2 , i.e., $\left(\frac{1-ay}{1+a} \right)^2$, which becomes

1) Cp. Memoirs, Kyushu, Vol. I, No. 3.

unity for $y = -1$. In the usual formula used for technical purposes, the coefficient runs as

$$a = \frac{\text{allowable normal stress } k_1}{\frac{K_1}{K_3} \times \text{allowable shearing stress } k_3}$$

instead of $\frac{1-ay}{1+a}$, where K_1 and K_3 are the statical strengths in tension and torsion, respectively.

Let us first assume that both the normal and shearing stresses vary between positive and negative stresses.

Then

$$k_3 = k_1 \frac{K_3}{K_1},$$

and

$$a = 1.$$

Next consider the case when the shearing stress varies between a maximum and zero, while the normal stress varies between positive and negative limits. Then

$$k_3 = 2k_1 \frac{K_3}{K_1},$$

and

$$a = \frac{1}{2}.$$

Lastly, if the shearing stress is steady, while the normal stress varies as in the foregoing cases, then

$$k_3 = 3k_1 \frac{K_3}{K_1},$$

and

$$a = \frac{1}{3}.$$

Such is the case in practical designing, the allowable stresses k_s 's for the above three typical cases, viz., $y = -1, 0, +1$, respectively, being in the ratios 1 : 2 : 3.

To compare the above two coefficients, take $a=\frac{1}{2}$ in the expression $\frac{1-ay}{1+a}$ corresponding to the stress ratios 1:1.5:3 as already stated instead of the usual practice above mentioned. Then we see that for

$$y=0, \quad \frac{1-ay}{1+a} = \frac{2}{3},$$

and for $y=1, \quad \frac{1-ay}{1+a} = \frac{1}{3}.$

As the present formulae (15) and (16) are of the same form as the technical ones, it is evident that the coefficient $\frac{1-ay}{1+a}$ plays the same rôle as a , and the discrepancy between the numerical values of these two coefficients is due to the different stress ratios taken in the calculation.

Testing Machine for Repeated Stress.

In order to test whether the inference drawn from the mathematical calculation as already laid down is true, some series of experiments were performed on steel specimens with the author's testing machine for repeated bending. This machine was designed on the same principle as that used by Eden, Rose, and Cunningham, (*loc. cit.*), the test piece being subject to the uniform bending moment and rotated continuously to repeat the varying normal stress thus induced in the cross section. But the construction being different, the following points should be particularly noted.

1) The tapered ends of the test piece are tightly fitted in the corresponding conical holes of the specimen holders by means of screws as shown in the diagrammatic sketch, Fig. 6, and the

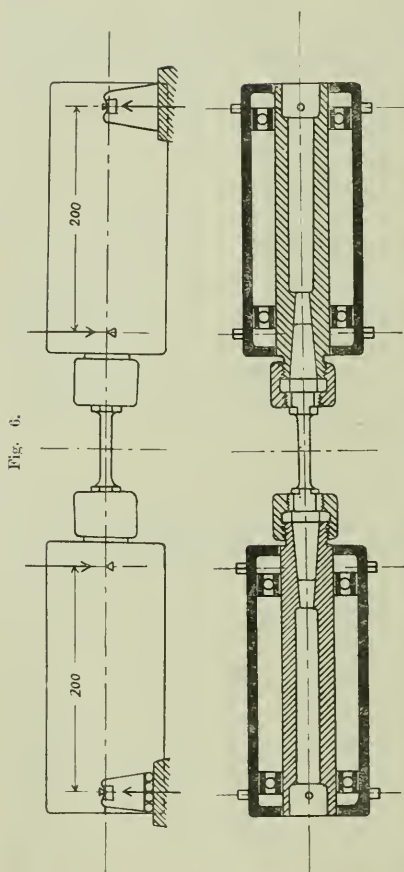


Fig. 6.

removal of the broken piece takes place by slackening the said joint. By this arrangement the trouble due to corrosion and wear of the contacting surfaces of the specimen and holders is minimized.

2) Each specimen holder is supported by two ball bearings in a casing, which rests in its position and carries the load by means of knife edges arranged in an exactly similar manner on the front and back sides of the casing. The distance between the two front (and two back) knife edges is 20 cm. in each casing, and the loading takes place equally. So the test piece fitted to the holders is subject to the uniform

bending moment along its middle part, and the amount of the bending moment can be accurately determined.

3) The support for the casing on the driving side of the machine rests on balls, and the driving shaft has a flexible coupling

in order to minimize the resistance for the slight movement of the casing due to the expansion or contraction of the specimen. By this means the extra stress on the test piece can be avoided.

By the above machine the normal stress in the rotating test piece was continuously varied between positive and negative stresses of equal magnitudes. Further the steady shearing stress was simultaneously induced in the specimen by connecting the testing machine with an electric absorption dynamometer, which consisted of a small D.C. dynamo placed on an iron frame suspended by ball bearings. The torque was measured by means of a weight placed on a scale pan hung from a knife edge 30 cm. distant from the axis of the shaft.

Materials for the Test.

The materials used in the present test were as follows:—

1) Low carbon ingot iron for series A. Carbon content = 0.078 %. The test pieces were taken from a round bar, 25 mm. in diameter. The material was cut in proper size, and it was heated in a gas furnace to 900° C. for 30 min. by enclosing it in a gas pipe. The cooling occurred naturally in the furnace. As to the structure of the material refer to the microphotograph, Fig. 7. The diameter of the test pieces was 6 mm. except two specimens, which were turned down to 5 mm. in order to induce possibly large shearing stress for the given amount of twisting moment.

2) Medium carbon ingot iron for series B. Carbon content = 0.304 %. The test pieces were taken from a round bar, 75 mm. in diameter. The material cut in proper size was heated in an electric furnace to 840° C. for one hour, and it was cooled down naturally in the furnace. The microphotograph is shown

in Fig. 8. The diameter of the test pieces was 6 mm. except a few ones, which were turned down to 4.5 mm. for the same purpose as in the series A.

3) Medium carbon ingot iron for series D. Carbon content = 0.251 %. The original bar was 75 mm. in diameter as in the series B. The material cut in proper size was heated in an electric furnace to 900° C. for 30 min., and then it was quenched in oil at 20° C. Subsequently it was reheated in another electric furnace to 650° C. for 30 min. The microphotograph, Fig. 9, shows the structure of the specimen thus treated. The majority of the pieces were 6 mm. in diameter, but five pieces were turned to 4.5 mm.

Results of Experiment.

In each series of test the number of the repetitions of alternating normal stress was measured both in the presence and absence of the simultaneously acting shearing stress, and the result of the experiment is shown in a diagram, in which the vertical distance represents the normal stress due to bending, and the horizontal distance gives the number of repetitions before fracture. The points plotted in the diagram are now united in a group by drawing a curve in proper position. In this case the experimental results for zero shearing stress were considered in one group, while those corresponding to equal, or nearly equal values of the shearing stress were gathered in another group. In this way the diagrams for the series A, B, and D were constructed as shown in Figs. 10, 11, and 12,¹⁾ respectively.

1) Explanation of symbols used in Figs. 10, 11, and 12.

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Fig. 7.

Series A



× 150

Fig. 8.

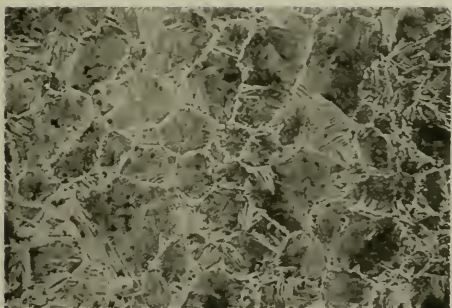
Series B



× 150

Fig. 9.

Series D



× 150

According to the results obtained from the series *A* and *B*, we learn that the greater part of the curves for certain values of τ lies above those for zero shearing stress; so it appears that the material can stand greater normal stress in the presence of τ than in its absence. Further the result of series *D* shows no considerable influence of τ on the endurance of the material. It is to be noted that the capacity of the dynamometer was so small that the shearing stress could not be made sufficiently large, although the diameter of the test piece was made possibly small in some cases, and therefore the maximum limit of the steady shearing stress, which does not produce any bad influence on the endurance, can not be given.

As to the broken specimens we may add that the fracture occurred in most cases along the cross sectional plane near the end of the cylindrical part of the piece, and no remarkable difference was observed between a specimen subject to no torque and that under the said action. Further, if the fracture at the end hastened the breaking, the influence appears insignificant, as the result is not discordant with that obtained from a piece broken in the middle part.

	Broken	Unbroken	Remark
τ zero	○ ✕	○→	✕ and ● refer to the specimens previously loaded in smaller stress range.
τ acting	● ●	●→	

Summary.

(1) First it was provisionally taken for granted on the ground of the experimental data already given by several authors, that the shearing stress and the tensile stress hypotheses are the two extreme laws governing the fatigue resistance of materials. Then taking a certain formula for the fatigue resistance, a calculation was made to find how the relative resistance varies according to the position of a plane considered in a rod subject to the combined normal and shearing stresses.

(2) As already made known by previous writers on a different basis, a shaft designed to endure a certain number of repetitions of normal stress varying between positive and negative limits of equal magnitudes was found to be strong enough to stand the same number of repetitions, if torsion is simultaneously allowed to act steadily, provided that the shearing stress then induced does not surpass a certain limit.

(3) Next a digression from the main subject was made for introducing simple formulae, which are of the same form as the usual formulae deduced by the statical calculation.

(4) Lastly, the experiment made for the purpose of elucidating the result of the calculation as stated in (2) was reported. In the experiment a special testing machine was used for repeating the bending stress by rotating the test piece subject to uniform bending moment. In three series of specimens tested in the above machine both under no torque, and under the simultaneous action of torque applied by driving a dynamo, it was proved, though not very exactly, that the endurance of material under alternating bending is independent of the action of torsion within the limits of the present experiment.

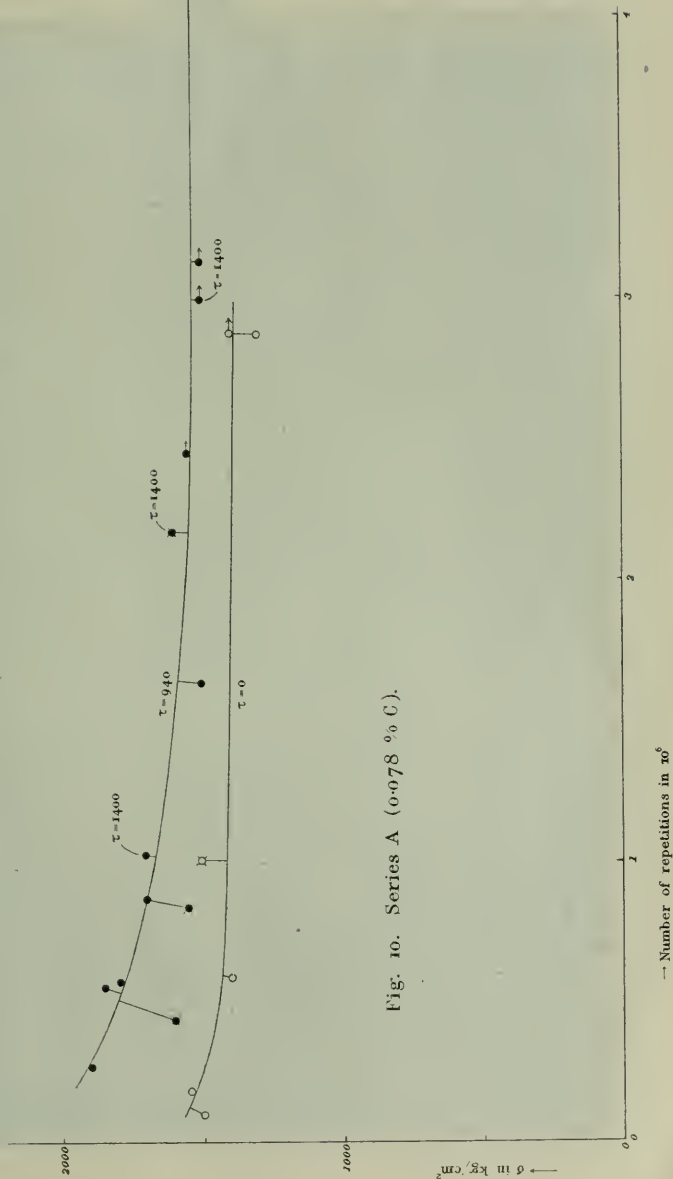


Fig. 10. Series A (0.078 % C).

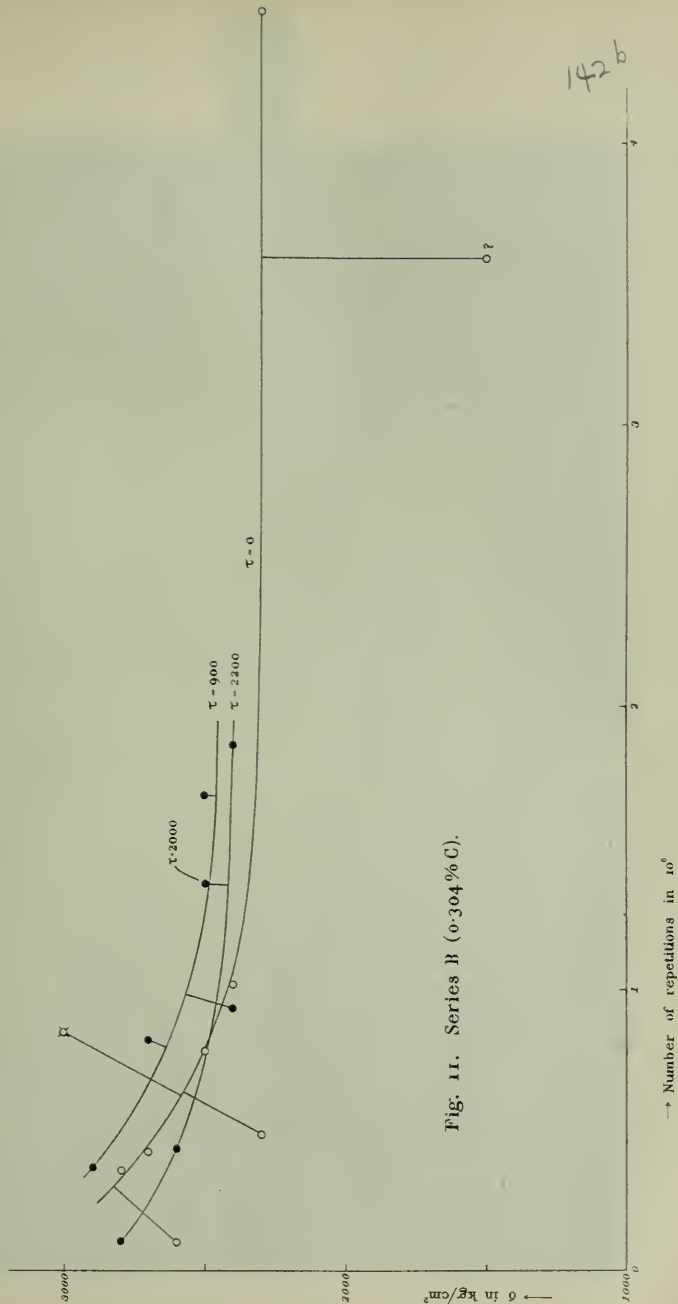


Fig. 11. Series B (0.304% C).

→ Number of repetitions in 10^6

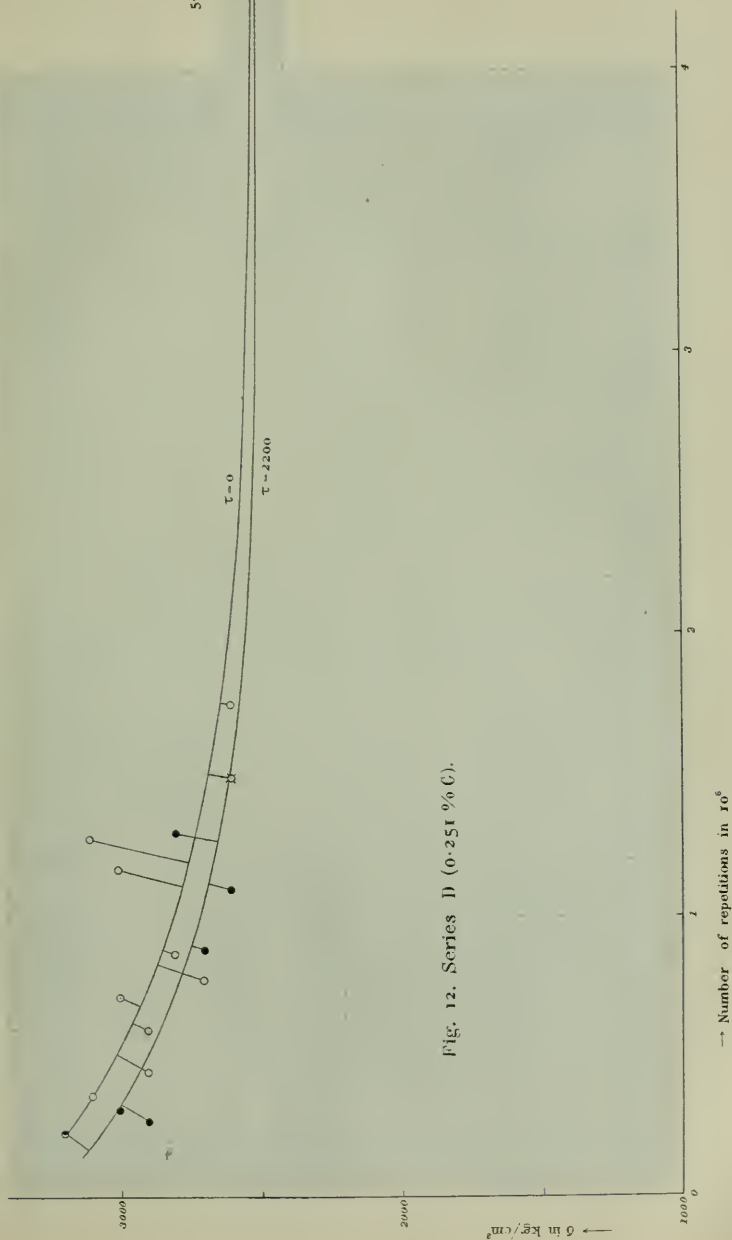


Fig. 12. Series D (0.251 % C).

Comparative Tests on Differently Inclined Buckets of a Pelton Wheel.

By

Junzo Nakahara, *M. E., Kogakuhakushi.*

(Received January 26th, 1921)

Object of the tests.

In fixing the buckets on the disc of a Pelton wheel, the central knife-edged ridges are not usually placed radially, but inclined backwards from the direction of rotation at such an angle as to obtain the most efficient working of an impinging jet. This inclination may be specified, where the ridges are straight, by an angle (β in Fig. 4) between the ridge line and the radius drawn to the tip of the ridge.

Authorities on this subject differ considerably in their opinions as to the best value of the angle β . Prof. A. H. Gibson states that the ridges are inclined at such an angle as to give normal incidence on the first impact of the jet¹⁾. On the other hand, some German authors as prof. R. Thomann recommends to choose such an angle as to make the ridge stand at right angles to the axis of the jet when the bucket occupies the mean position between the positions at which the full section of the jet begins and ends to strike the buckets²⁾; whilst Prof. Wagenbach, Prof. Camerer, and others propose such an angle that the

1) Gibson "Hydraulics and its Applications," p. 445, 1919.

2) Thomann "Wasser-Turbinen," p. 217, 1908.

central ridge shall stand at right angles to the direction of the relative entrance velocity of the jet at the same mean position of the bucket¹. The object of the tests is to determine directly which inclination of the buckets does give the best efficiency.

Conditions of the tests.

The Pelton wheel used for the tests was of a Pelton-Doble type manufactured by Böving and Co., London, installed in the Hydraulic Laboratory of the Engineering Division, Kyushu University (Fig. 1). It is 520 mm. in diameter and designed

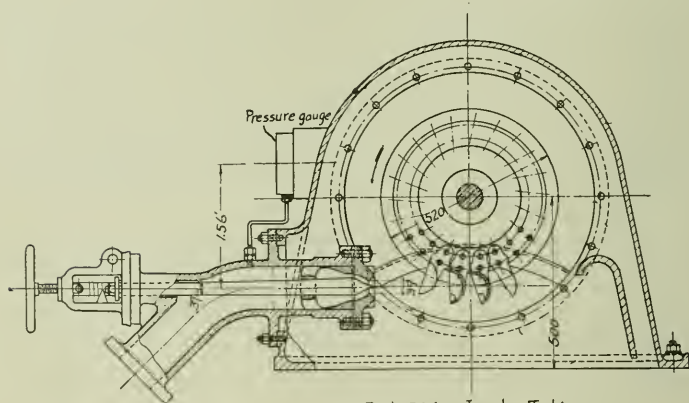


Fig. 1 Böving Impulse Turbine.

to develop 6 B. H. P. under a head of 30 meters at 400 r.p.m. To reduce the bearing friction to a minimum, the shaft is supported on two ball bearings fixed to the covers. The Doble ellipsoidal buckets 22 in number were specially made for this

1) Camere "Wasserkraftmaschinen, p. 555, 1914. Hütte II, p. 311, 1911.

test of exactly the same form as the original ones but the two laps for fixing them on the disc, were made broader to enable the inclination of the buckets to be varied as required, keeping the tip radius constant (Fig. 2).

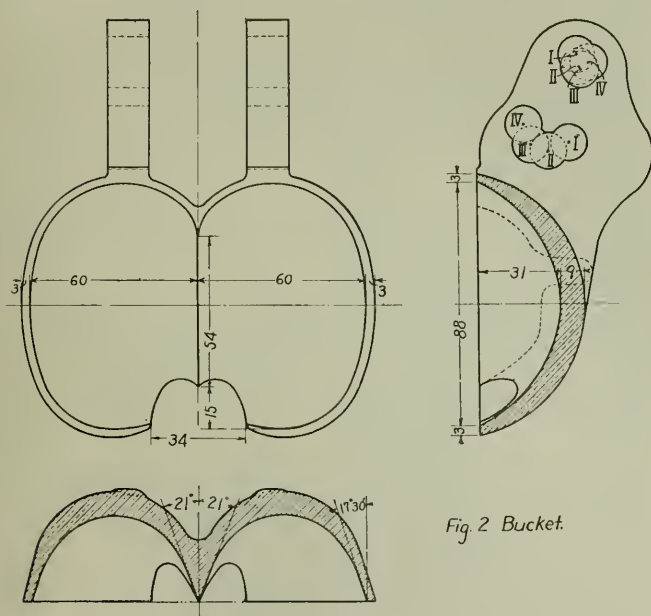


Fig. 2 Bucket.

The water was supplied by a 125 mm. two stage turbo-pump through a high pressure tank 6 feet in diameter and 32 feet 7 inches long, by means of which we were able to maintain the head at 100 feet almost constant during the test. The pressure head was measured by a Bourdon gauge fixed just before the nozzle entrance, and the discharge from the wheel was

measured by a rectangular notch with end contractions, 20 inches wide, located on the end of a discharge channel $4\frac{1}{2}$ feet wide and $3\frac{1}{2}$ feet deep. The power was measured by a Prony brake specially made for this wheel, and the speed by an integrating revolution counter directly attached to the shaft. All these apparatus, the weir, gauge and brake were accurately calibrated beforehand at the Laboratory, and they worked quite satisfactorily during the whole course of the experiment. Fig. 3 is the photo of the Böving impulse turbine showing the arrangement of the brake wheel, pressure gauge &c.

Experiments were made in this way with four different inclinations of buckets, which were chosen of such angles that the dividing ridge stood at:

Case I —Right angles to the jet at the end of full jet.

Case II —Right angles to the jet at the mean position
between the beginning and end of full jet.

Case III—Right angles to the jet at the beginning of
full jet.

Case IV—Right angles to the relative entrance velocity
of the mid particles of jet at the same mean
position of a bucket as in the Case II.

The values of β angle which fulfil the conditions stated above may be determined analytically in the following way.

Notations:

r = wheel radius or the perpendicular let fall from
the wheel centre on the axis of jet,

r_a = radius of the tip of the dividing ridge of a bucket,

u, u_a = circumferential linear velocities at r and r_a respectively,

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Fig. 3.



θ = pitch angle of buckets,

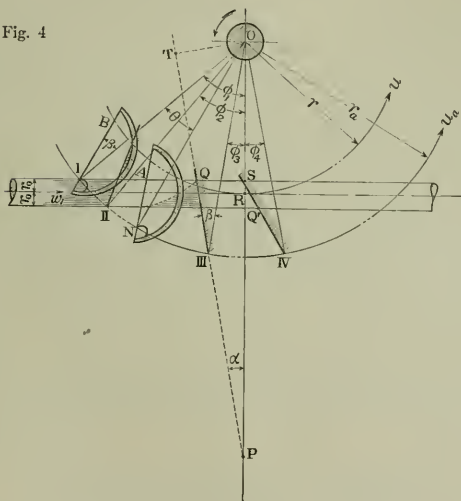
$\phi_{(1...4)}$ = angle between the tip radius of a bucket in any position and the radius perpendicular to the axis of the jet, representing the particular positions of a bucket by numerical suffixes,

β = angle between the central ridge of the bucket and the tip radius,

w_1 = absolute velocity of the jet,

r_o = radius of the jet.

Fig. 4



Let A and B in Fig. 4 represent the two consecutive buckets whose tip radii subtend a pitch angle θ at the centre O . Now as the wheel turns on, the bucket A will come first before the jet and the first drop will strike the bucket A when it arrives at the position I (shown in the figure as occupied by

the bucket *B*.) The tip radius will, then, make an angle ϕ_1 with the radius *OR* perpendicular to the jet axis. As the wheel turns further on, the rate of inflow of water in-to the bucket will increase more and more, till it arrives at the position *II* where the full section of the jet will begin to strike the bucket, and then the tip radius makes an angle ϕ_2 with the radius *OR*. The angles ϕ_1 and ϕ_2 are given by the relations:

$$\phi_1 = \cos^{-1} \frac{r-r_o}{r_a}; \quad \phi_2 = \cos^{-1} \frac{r+r_o}{r_a}$$

Then the full jet will continue to strike the bucket *A* until the first particle of the jet cut off by the tip of central ridge of the succeeding bucket *B* at *I* reaches the ridge of *A* as at *Q*. This position of the bucket *A* denoted by ϕ_3 can be found by the consideration that the time taken by the first particle of the water cut off by *B* to traverse a distance *IQ* is equal to that taken by the bucket *A* to turn through the angle $NOIII = \phi_1 - \theta - \phi_3$. Produce the ridge *IIIQ* both ways to meet *OR* produced at *P*, and the perpendicular from *O* on *IIIQ* at *T*. Then

$$IQ = IS - QS = r_a \sin \phi_1 - PS \tan \alpha, \text{ where } \alpha = \beta - \phi_3. \text{ And} \\ PS = PO - SO = OT \operatorname{cosec} \alpha - (r - r_o) = r_a \sin \beta \operatorname{cosec} \alpha - r + r_o.$$

Hence we have

$$\frac{r_a \sin \phi_1 - (r_a \sin \beta \operatorname{cosec} \alpha - r + r_o) \tan \alpha}{w_1} = \frac{(\phi_1 - \theta - \phi_3) r_a}{u_a}$$

which may be transformed in-to

$$A \alpha - B \tan \alpha + C \sec \alpha - D = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Where

$$\alpha = \beta - \phi_3.$$

$$A = \frac{w_1}{u} \cdot \frac{r}{r_a}; \quad B = \cos \phi_1; \quad C = \sin \beta; \quad D = \sin \phi_1 - A(\phi_1 - \theta - \beta)$$

As the wheel turns on, only a portion of the full jet will strike the bucket A , the rate of inflow decreasing more and more until a position ϕ_4 is reached; here the last drop of water impinges upon the bucket as at Q' , and the angle ϕ_4 may be determined in a similar way to the angle ϕ_3 . The equation is exactly of the same form, namely

$$A \alpha - B \tan \alpha + C \sec \alpha - D = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Where

$$\alpha = \beta + \phi_4.$$

$$A = \frac{w_1}{u} \cdot \frac{r}{r_a}; \quad B = \cos \phi_2; \quad C = \sin \beta; \quad D = \sin \phi_2 - A(\phi_2 - \theta - \beta)$$

The total range of the incidence of jet may thus be divided into three parts, $\phi_1 - \phi_2$, $\phi_2 - \phi_3$ and $\phi_3 + \phi_4$, of which the middle part during which the full jet impinges upon the bucket is the most important, and it will naturally follow that, if high efficiency is to be expected, we should so dispose the buckets as to obtain the most favourable working of the jet during this part.

Now in order that the conditions stated in the four cases above mentioned shall be fulfilled, the angles must have the following relations

$$\text{Case I : } \beta = \phi_3$$

$$\text{Case II : } \beta = \frac{1}{2}(\phi_2 + \phi_3)$$

$$\text{Case III : } \beta = \phi_2$$

$$\text{Case IV : } \cot \beta = (\cos \delta - \frac{u}{w_1} \cdot \frac{r_a}{r}) \operatorname{cosec} \delta$$

$$\text{where } \delta = \frac{1}{2}(\phi_2 + \phi_3).$$

$$\frac{u_a}{w_1} = \frac{\sin(\beta - \delta)}{\sin \beta},$$

which can easily be transformed into the form

$$\cot \beta = (\cos \delta - \frac{u}{w_1} \cdot \frac{r_a}{r}) \operatorname{cosec} \delta$$

Putting these four relations into the equation (1) separately, we can find the inclinations (β) of the ridge corresponding to the four cases. The data for the calculations are as follows:

Radius of wheel $r = 260 \text{ mm.}$

Radius of bucket's tip $r_a = 298 \text{ mm.}$

Radius of jet $r_o = 16 \text{ mm.}$

Ratio of peripheral velocity to

velocity of jet $\frac{u}{w_1} = 0.4525$

Normal speed 400 r. p. m.

The results of the calculation are given in Table I.

TABLE I.

Case: I	$\phi_1 = 35^\circ 2'$	$\phi_2 = 22^\circ 10'$	$\phi_3 = 3^\circ 18'$	$\phi_4 = 10^\circ 19'$	$\beta = 5^\circ 18'$
Case: II	„	„	$4^\circ 58'$	$9^\circ 36'$	$13^\circ 34'$
Case: III	„	„	$6^\circ 24'$	$8^\circ 55'$	$22^\circ 10'$
Case: IV	„	„	$7^\circ 34'$	$8^\circ 07'$	$29^\circ 48'$

Accordingly, the bolt holes of the bucket laps were newly drilled out in each case and the buckets carefully fixed with the inclination of the ridge equal to the corresponding β in the Table I.

Results of Tests.

A series of complete brake tests was made in each case for five different strokes 0,188—0,400—0,650—0,808—1,000 inches, corresponding to 1,13—2,4—3,9—4,85—6 turns of the needle, the outer portion of which is screwed with a square thread of $\frac{1}{6}$ inch pitch, and provided with a hand wheel graduated in hundredth of a revolution. The head was maintained nearly constant at 100 feet while the speed varied; readings of the speed counter, weir, and brake weights were taken for 20 to 44 different speeds according to circumstances, each way first decreasing the brake load and then increasing the same. The results of the tests are given on Tables II—XXI and plotted on Fig. 6—10. The curves for the four cases are drawn on the same plate for comparison. In plotting the curves, the speed readings have been reduced to a standard head 100 feet, assuming that the efficiency will remain constant for a small variation of the head and the speed vary as the square root of the head.

The maximum efficiencies thus obtained in the four cases are given on Table XXII for comparison, and plotted in Fig. 15 taking the discharge as abscissa.

In order to show all the relations between the speed, discharge, and efficiency for each of the four cases in one diagram, the characteristic curves were indirectly plotted from the preceding sets of curves. They are shown in Fig. 11—14.

TABLE II.

Brake test of Böving impulse turbine.

Case 1; needle stroke 0,188 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Dis- charge per second q cubic ft.	Brake load lbs.	Total avail- able head h ft.	Total avail- able H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
11th. Aug. 1920.	1	4	50,3	0,00252	0,167	15,93	100,06	1,899	0,326	17,2	50,3
	2	4	93,5	"	0,167	14,82	99,86	1,895	0,565	29,8	93,6
	3	3	166,3	"	0,168	13,72	99,86	1,907	0,929	48,8	166,4
	4	4	216,5	"	0,168	12,62	99,86	1,907	1,113	58,4	216,7
	5	3	268,3	"	0,168	11,52	99,76	1,905	1,260	66,1	268,6
	6	3	310,0	"	0,167	10,42	99,76	1,893	1,316	69,5	310,4
	7	3	352,3	"	0,168	9,31	99,86	1,907	1,337	70,1	352,6
	8	4	407,0	"	0,170	8,21	100,16	1,935	1,362	70,4	406,7
	9	3	447,7	"	0,170	7,11	99,86	1,929	1,297	67,2	448,0
	10	3	479,0	"	0,170	6,01	99,76	1,927	1,172	60,8	479,6
	11	3	443,0	"	0,170	7,11	99,86	1,929	1,283	66,5	443,3
	12	3	400,0	"	0,170	8,21	99,96	1,931	1,338	69,3	400,1
	13	4	352,8	"	0,170	9,31	99,96	1,931	1,339	69,3	352,9
	14	3	316,0	"	0,170	10,42	99,76	1,927	1,342	69,6	316,4
	15	4	272,8	"	0,170	11,52	100,36	1,939	1,281	66,1	372,3
	16	4	214,3	"	0,170	12,62	99,66	1,925	1,102	57,3	214,7
	17	3	162,0	"	0,170	13,72	100,16	1,935	0,906	46,8	161,9
	18	3	95,3	"	0,168	14,82	99,66	1,903	0,576	30,2	95,5
	19	3	57,7	"	0,170	15,93	100,36	1,939	0,375	19,3	57,5
13th.	20	4	655,0	"	0,165	0	99,86	1,873	0	0	655,4

Temperature of water=75° F. Specific weight of water=62,5 lbs.

TABLE III.

Brake test of Böwing impulse turbine.

Case 1; needle stroke 0,400 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Dis- charge per second q cubic ft.	Brake load lbs.	Total avail- able head h ft.	Total avail- able H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
10th. Aug. 1920.	1	2	132,0	0,00482	0,342	31,36	99,89	3,882	1,687	43,5	132,1
	2	3	183,7	"	0,345	29,15	99,99	3,920	2,182	55,6	183,7
	3	3	227,0	"	0,344	26,95	99,49	3,889	2,493	64,1	227,6
	4	4	278,8	"	0,344	24,74	99,49	3,889	2,810	72,3	279,5
	5	4	335,5	"	0,344	22,54	99,59	3,893	3,081	79,1	336,2
	6	4	358,0	"	0,346	21,43	100,09	3,935	3,126	79,4	357,9
	7	3	375,3	"	0,346	20,33	99,89	3,928	3,112	79,2	375,5
	8	3	394,7	"	0,346	19,23	99,59	3,916	3,093	79,0	395,5
	9	3	417,3	"	0,346	18,19	99,59	3,916	3,093	79,0	418,1
	10	3	438,7	"	0,346	17,03	99,69	3,920	3,044	77,6	439,4
	11	3	453,7	"	0,348	15,93	99,59	3,939	2,945	74,8	454,6
	12	3	471,0	"	0,347	14,82	99,69	3,931	2,844	72,4	471,7
	13	3	491,0	"	0,346	13,72	99,59	3,916	2,745	70,1	492,0
	14	3	527,7	"	0,347	11,52	99,59	3,927	2,477	63,1	528,8
	15	3	561,3	"	0,348	9,31	99,49	3,934	2,130	54,2	562,8
	16	3	594,0	"	0,348	7,11	99,69	3,942	1,721	43,7	594,9
	17	3	561,0	"	0,348	9,31	99,69	3,942	2,129	54,0	561,9
	18	3	526,7	"	0,347	11,52	99,59	3,927	2,472	63,0	527,8
	19	3	488,0	"	0,347	13,72	99,59	3,927	2,728	69,5	489,0
	20	3	473,3	"	0,348	14,82	99,69	3,942	2,858	72,5	474,0
	21	3	453,7	"	0,348	15,93	99,59	3,939	2,945	74,8	454,6
	22	3	438,0	"	0,346	17,03	99,59	3,916	3,039	77,6	438,9
	23	3	419,0	"	0,346	18,19	99,69	3,920	3,106	79,2	419,6
	24	3	397,7	"	0,348	19,23	99,69	3,942	3,116	79,1	398,3
	25	3	379,7	"	0,348	20,33	99,89	3,950	3,148	79,7	380,0
	26	3	353,7	"	0,349	21,43	99,49	3,946	3,089	78,3	354,6
	27	3	328,7	"	0,347	22,54	99,59	3,927	3,019	76,9	329,4
	28	3	285,7	"	0,346	24,74	99,99	3,931	2,880	73,3	285,7
	29	3	254,3	"	0,346	26,95	99,89	3,928	2,573	65,5	254,4
	30	3	186,0	"	0,346	29,15	100,29	3,943	2,209	56,0	185,7
	31	2	124,0	"	0,346	31,36	99,89	3,928	1,584	40,3	124,1
13th.	32	3	694,3	"	0,340	0	99,69	3,855	0	0	695,1

Temperature of water=74° F. Specific weight of water=62,5 lbs.

TABLE IV.

Brake test of Böving impulse turbine.

Case I; needle stroke 0,650 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second a cubic ft.	Brake load lbs.	Total available head ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head
13th. Aug. 1920.	1	3	148,3	0,00717	0,533	46,79	99,97	6,056	2,823	46,7	148,3
	2	3	181,3	"	0,533	44,58	99,97	6,056	3,293	54,4	181,3
	3	3	218,0	"	0,533	42,38	99,97	6,056	3,765	62,2	218,0
	4	3	252,7	"	0,533	40,18	99,77	6,044	4,134	68,5	253,0
	5	3	285,7	"	0,534	37,97	99,87	6,061	4,420	72,9	285,9
	6	3	319,3	"	0,535	35,77	99,77	6,067	4,654	76,7	319,7
	7	3	351,7	"	0,535	33,56	99,97	6,078	4,810	79,0	351,8
	8	3	376,0	"	0,535	31,36	99,87	6,072	4,805	79,1	376,5
	9	3	406,0	"	0,534	29,15	99,87	6,061	4,823	79,6	406,5
	10	3	428,3	"	0,535	26,95	99,77	6,067	4,704	77,5	428,8
	11	3	456,3	"	0,535	24,74	100,27	6,097	4,600	75,5	455,8
	12	3	478,7	"	0,535	22,54	100,17	6,091	4,396	72,2	478,3
	13	3	501,0	"	0,535	20,33	99,87	6,072	4,150	68,4	501,3
	14	3	524,7	"	0,534	18,19	99,87	6,061	3,889	64,2	525,0
	15	3	551,7	"	0,535	15,93	99,77	6,067	3,581	59,0	552,4
	16	3	575,0	"	0,535	13,72	99,67	6,060	3,214	53,0	575,9
	17	3	596,3	"	0,535	11,52	99,87	6,072	2,800	46,1	596,8
	18	3	616,3	"	0,535	9,31	100,17	6,091	2,339	38,4	615,8
	19	3	640,7	"	0,535	7,11	100,07	6,084	1,896	31,2	640,4
	20	3	616,7	"	0,535	9,31	99,67	6,060	2,340	38,6	617,7
	21	3	596,7	"	0,534	11,52	99,87	6,061	2,802	46,2	596,7
	22	4	573,5	"	0,535	13,72	100,07	6,084	3,206	52,7	573,3
	23	3	552,7	"	0,536	15,93	100,07	6,096	3,588	58,9	552,5
	24	3	525,7	"	0,535	18,19	99,77	6,067	3,897	64,2	526,3
	25	3	495,0	"	0,535	20,33	99,77	6,067	4,101	67,6	495,6
	26	4	476,0	"	0,535	22,54	99,77	6,067	4,371	72,1	476,6
	27	3	452,3	"	0,535	24,74	99,77	6,067	4,560	75,2	452,8
	28	3	428,7	"	0,535	26,95	99,87	6,072	4,708	77,5	429,0
	29	3	403,3	"	0,535	29,15	99,97	6,078	4,791	78,8	403,4
	30	4	377,0	"	0,537	31,36	99,97	6,101	4,817	79,0	377,1
	31	3	353,3	"	0,535	33,56	100,07	6,084	4,832	79,4	353,2
	32	3	322,0	"	0,536	35,77	100,17	6,102	4,693	76,9	321,7
	33	3	287,0	"	0,535	37,97	99,87	6,072	4,440	73,1	287,2
	34	3	252,0	"	0,535	40,18	99,97	6,078	4,093	67,4	252,0
	35	3	217,7	"	0,536	42,38	99,87	6,084	3,760	61,8	217,8
	36	3	183,7	"	0,536	44,58	99,77	6,078	3,337	54,9	183,9
	37	3	146,3	"	1,534	46,79	99,87	6,061	2,790	46,1	146,4
13th.	38	4	712,5	"	0,514	0	100,07	5,845	0	0	712,2

Temperature of water=77° F. Specific weight of water=62,5 lbs.

TABLE V.

Brake test of Böving impulse turbine.

Case I; needle stroke, 0,808 in. Brake arm 2,14 ft.

Date	No.	Dura- tion of test min.	R.P.M.	Area of nozzle opening sq. ft.	Dis- charge per second cubic ft.	Brake load lbs.	Total avail- able head h ft.	Total avail- able H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
11th. Aug. 1920.	1	3	153,0	0,00857	0,599	53,41	99,75	6,790	3,329	49,0	153,2
	2	3	187,3	"	0,599	51,20	100,05	6,811	3,907	57,4	187,3
	3	3	218,7	"	0,600	48,99	99,95	6,815	4,265	64,1	218,7
	4	3	249,7	"	0,602	46,79	99,96	6,838	4,761	69,6	249,7
	5	3	279,0	"	0,603	44,58	99,96	6,850	5,068	74,0	279,1
	6	3	302,0	"	0,601	42,38	99,96	6,821	5,215	76,5	302,4
	7	3	333,3	"	0,602	40,18	99,96	6,838	5,454	79,8	333,1
	8	3	357,0	"	0,600	37,97	99,85	6,808	5,523	81,1	357,5
	9	3	382,3	"	0,602	35,77	99,96	6,838	5,572	81,5	382,4
	10	3	405,0	"	0,603	33,56	100,26	6,870	5,538	80,6	404,6
	11	3	424,7	"	0,602	31,36	99,76	6,825	5,427	79,5	423,2
	12	3	446,3	"	0,602	29,15	99,96	6,838	5,301	77,5	446,4
	13	3	466,7	"	0,603	26,95	100,06	6,856	5,125	74,8	466,6
	14	3	491,3	"	0,603	24,74	100,26	6,870	4,953	72,1	490,8
	15	3	505,7	"	0,611	22,54	99,86	6,821	4,644	68,1	506,1
	16	3	537,7	"	0,604	20,33	100,16	6,875	4,454	64,8	537,3
	17	3	556,0	"	0,604	18,19	100,16	6,875	4,121	59,9	555,6
	18	3	573,7	"	0,605	15,93	100,06	6,879	3,724	54,1	573,6
	19	3	598,0	"	0,605	13,72	100,26	6,893	3,343	48,5	597,4
	20	3	614,7	"	0,605	11,52	100,26	6,893	2,886	41,9	614,1
	21	3	634,3	"	0,605	9,31	99,86	6,865	2,407	35,1	634,8
	22	4	664,3	"	0,605	7,11	99,86	6,865	1,924	28,0	664,7
	23	4	632,8	"	0,604	9,31	99,86	6,854	2,401	35,0	633,2
	24	3	610,3	"	0,604	11,52	99,96	6,861	2,865	41,8	610,4
	25	4	595,5	"	0,606	13,72	100,36	6,911	3,329	48,2	594,4
	26	3	578,3	"	0,606	15,93	100,36	6,911	3,754	54,3	577,3
	27	4	556,5	"	0,604	18,19	99,96	6,861	4,125	60,1	556,6
	28	3	529,3	"	0,604	20,33	99,76	6,848	4,384	64,0	529,8
	29	3	512,0	"	0,604	22,54	99,96	6,861	4,702	68,5	512,1
	30	3	492,3	"	0,606	24,74	100,26	6,904	4,963	71,9	491,8
	31	3	473,7	"	0,606	26,95	100,16	6,897	5,202	75,4	473,4
	32	4	459,5	"	0,605	29,15	99,86	6,865	5,351	78,0	450,8
	33	3	430,3	"	0,606	31,36	100,06	6,890	5,498	79,8	430,3
	34	3	410,0	"	0,606	33,56	100,06	6,890	5,606	81,4	409,9
	35	3	390,7	"	0,606	35,77	99,96	6,884	5,695	82,7	390,8
	36	3	367,7	"	0,606	37,97	99,86	6,876	5,689	82,7	367,9
	37	3	337,0	"	0,606	40,18	99,96	6,884	5,517	80,2	337,1
	38	3	310,3	"	0,604	42,38	99,76	6,848	5,359	78,3	310,7
	39	3	283,7	"	0,604	44,58	99,96	6,861	5,154	75,1	283,8
	40	3	256,7	"	0,604	46,79	99,96	6,861	4,895	71,4	256,7
	41	3	226,0	"	0,604	48,99	99,86	6,854	4,512	65,8	226,0
	42	3	199,0	"	0,603	51,20	100,06	6,856	4,151	60,6	199,0
	43	3	164,0	"	0,603	53,41	99,96	6,850	3,569	52,1	164,0
13th.	44	4	728,8	"	0,600	0	100,06	6,823	0	0	728,6

Temperature of water = 75° F. Specific weight of water = 62,5 lbs.

TABLE VI.

Brake test on Böving impulse turbine.

Case I; needle stroke 1,000 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second cu. ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
12th. Aug. 1920.	1	3	198.3	0.01017	0.690	57.82	99.97	7.839	4.674	59.6	198.3
	2	3	242.3	"	0.688	54.51	100.27	7.840	5.382	68.7	242.1
	3	3	279.0	"	0.689	51.20	99.97	7.828	5.820	74.4	279.0
	4	4	298.3	"	0.688	48.99	99.87	7.809	5.952	76.2	298.5
	5	3	325.3	"	0.687	46.79	99.97	7.805	6.202	79.5	325.3
	6	3	353.0	"	0.689	44.58	100.37	7.858	6.412	81.6	352.4
	7	4	368.0	"	0.688	42.38	99.97	7.816	6.356	81.3	368.0
	8	3	389.3	"	0.687	40.18	100.07	7.812	6.373	81.6	389.2
	9	4	406.0	"	0.688	37.97	100.07	7.824	6.281	80.3	405.8
	10	3	432.3	"	0.688	35.77	100.37	7.846	6.301	80.3	431.6
	11	3	447.0	"	0.689	33.56	100.07	7.835	6.113	78.0	446.8
	12	3	466.3	"	0.688	31.36	100.17	7.832	5.958	73.3	465.9
	13	3	482.7	"	0.688	29.15	100.07	7.824	5.733	73.2	482.5
	14	3	502.0	"	0.690	26.95	100.07	7.846	5.513	70.3	501.8
	15	3	527.0	"	0.688	23.64	99.87	7.809	5.077	65.0	527.3
	16	3	557.7	"	0.690	20.33	100.37	7.870	4.621	58.7	556.8
	17	4	582.5	"	0.690	17.03	100.07	7.846	4.042	51.5	582.3
	18	4	611.3	"	0.690	13.72	100.07	7.846	3.417	43.6	611.1
	19	3	641.7	"	0.690	10.42	100.27	7.862	2.725	34.7	641.1
	20	3	678.7	"	0.690	7.11	100.07	7.846	1.966	25.1	678.4
13th.	21	3	645.0	"	0.690	10.42	100.07	7.846	2.738	34.9	644.8
	22	3	608.7	"	0.688	12.72	99.97	7.816	3.406	43.6	608.8
	23	3	582.3	"	0.689	17.03	99.87	7.820	4.041	51.7	582.7
	24	3	555.3	"	0.688	20.33	99.97	7.816	4.601	58.9	555.4
	25	3	526.7	"	0.688	23.64	99.97	7.816	5.074	64.9	526.8
	26	3	504.3	"	0.690	26.95	100.67	7.894	5.538	70.2	503.2
	27	3	483.7	"	0.688	29.15	100.07	7.824	5.745	73.4	483.5
	28	3	466.0	"	0.690	31.36	100.37	7.870	5.954	75.7	465.1
	29	3	446.7	"	0.691	33.56	100.07	7.847	6.108	77.8	446.5
	30	3	432.0	"	0.691	35.77	100.27	7.863	6.297	80.1	431.6
	31	3	411.3	"	0.690	37.97	100.07	7.846	6.264	81.1	411.1
	32	3	391.7	"	0.690	40.18	99.97	7.839	6.412	81.8	391.8
	33	3	370.3	"	0.690	42.38	100.37	7.870	6.396	81.2	369.6
	34	3	348.3	"	0.691	44.58	99.97	7.840	6.327	80.7	348.4
	35	3	329.0	"	0.690	46.79	100.07	7.846	6.272	79.9	328.9
	36	3	304.3	"	0.690	48.99	100.07	7.846	6.074	77.4	304.2
	37	3	280.7	"	0.690	51.20	100.47	7.878	5.856	74.3	280.1
	38	3	245.0	"	0.691	54.51	100.47	7.880	5.442	69.1	244.4
	39	2	203.5	"	0.688	57.82	100.07	7.824	4.794	61.3	203.1
	40	4	732.5	"	0.684	0	100.07	7.778	0	0	732.2

Temperature of water = 76° F. Specific weight of water = 62.5 lbs.

TABLE VII.

Brake test of Böving impulse turbine.

Case II; needle stroke 0,188 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second q cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
28th. July 1920.	1	2	52,0	0,0052	0,167	15,93	100,19	1,901	0,338	17,8	52,0
	2	3	78,3	"	0,167	14,82	99,99	1,898	0,473	24,9	78,3
	3	3	167,3	"	0,167	13,72	99,99	1,898	0,935	49,3	167,3
	4	3	215,7	"	0,167	12,62	100,29	1,903	1,109	58,3	215,4
	5	3	266,3	"	0,167	11,52	100,09	1,899	1,250	65,8	266,2
	6	4	310,7	"	0,168	10,42	100,09	1,911	1,316	68,9	310,6
	7	4	358,3	"	0,169	9,31	99,99	1,920	1,360	70,8	358,3
	8	4	403,8	"	0,169	8,21	99,99	1,920	1,351	70,4	403,7
	9	4	446,3	"	0,169	7,11	100,39	1,928	1,293	67,1	445,2
	10	4	484,0	"	0,169	6,01	99,99	1,920	1,185	61,7	484,0
	11	3	442,7	"	0,169	7,11	99,79	1,916	1,283	67,0	443,2
	12	3	400,0	"	0,169	8,21	100,19	1,924	1,338	69,5	399,6
	13	3	357,3	"	0,169	9,31	99,99	1,920	1,356	70,6	357,3
	14	4	314,3	"	0,169	10,42	99,99	1,920	1,334	69,5	314,3
	15	3	267,3	"	0,169	11,52	99,99	1,920	1,255	65,4	267,3
	16	3	216,0	"	0,169	12,62	99,79	1,916	1,111	58,0	216,0
	17	3	163,0	"	0,169	13,72	99,99	1,920	0,911	47,5	163,0
	18	3	88,3	"	0,169	14,82	99,99	1,920	0,533	27,8	88,3
	19	2	57,5	"	0,169	15,93	100,39	1,928	0,373	19,4	53,4
31st.	20	4	655,0	"	0,167	0	100,09	1,899	0	0	654,7

Temperature of water = 74° F. Specific weight of water = 62,5 lbs.

TABLE VIII.

Brake test of Böving impulse turbine.

Case II; needle stroke 0,400 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 10 ft. head.
28th. July 1920.	1	3	132,0	0,00482	0,340	31,36	100,20	3,871	1,687	43,6	131,9
	2	3	177,0	"	0,347	29,15	100,30	3,955	2,102	53,2	176,8
	3	3	273,0	"	0,345	26,95	100,40	3,936	2,602	66,1	236,5
	4	3	283,7	"	0,345	24,74	99,70	3,908	2,861	73,2	284,1
	5	3	333,7	"	0,345	22,54	100,30	3,932	3,065	78,0	333,2
	6	3	355,3	"	0,345	21,43	100,50	3,938	3,103	78,8	354,5
	7	3	381,7	"	0,345	20,33	100,40	3,936	3,162	80,3	381,0
	8	3	401,3	"	0,345	19,23	100,10	3,924	3,144	80,1	401,1
	9	3	424,0	"	0,345	18,19	100,50	3,938	3,143	79,8	423,0
	10	3	449,0	"	0,345	17,03	100,10	3,924	3,053	77,8	439,8
	11	3	458,0	"	0,345	15,93	99,90	3,916	2,973	75,9	458,2
	12	3	480,7	"	0,345	14,82	100,10	3,924	2,903	74,0	480,5
	13	3	493,3	"	0,345	13,72	100,60	3,943	2,758	70,0	491,9
	14	3	530,3	"	0,345	11,52	100,60	3,943	2,489	63,1	523,7
	15	3	563,3	"	0,345	9,31	100,20	3,928	2,138	54,4	562,7
	16	4	600,0	"	0,350	7,11	100,40	3,993	1,738	43,5	598,8
	17	4	563,3	"	0,347	9,31	100,00	3,943	2,138	54,2	563,3
	18	4	533,7	"	0,345	11,52	100,20	3,928	2,505	63,8	533,2
	19	3	494,0	"	0,345	13,72	100,00	3,920	2,762	70,5	494,0
	20	3	480,0	"	0,345	14,82	100,60	3,943	2,899	73,5	478,6
	21	3	463,3	"	0,350	15,93	100,70	4,005	3,007	75,1	461,7
	22	3	442,0	"	0,350	17,03	100,60	4,001	3,067	76,7	441,0
	23	3	423,3	"	0,350	18,19	100,50	3,997	3,138	78,5	422,3
	24	3	401,7	"	0,350	19,23	100,40	3,993	3,147	78,8	400,9
	25	3	377,0	"	0,350	20,33	100,50	3,997	3,123	78,1	376,1
	26	3	357,5	"	0,350	21,43	100,10	3,981	3,122	78,4	357,4
	27	4	331,3	"	0,350	22,54	100,20	3,985	3,043	76,4	331,0
	28	3	284,7	"	0,345	24,74	100,10	3,924	2,871	73,2	284,6
	29	3	234,7	"	0,345	26,95	100,20	3,928	2,577	63,6	234,5
	30	3	183,3	"	0,345	29,15	100,40	3,936	2,177	55,3	182,9
	31	3	109,7	"	0,345	31,36	100,10	3,924	1,402	35,7	109,7
31st.	32	3	704,3	"	0,342	0	100,30	3,898	0	0	704,0

Temperature of water=74° F. Specific weight of water=62,5 lbs.

TABLE IX.

Brake test of Böving impulse turbine.

Case II; needle stroke 0,650 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
29th. July 1920.	1	3	126,0	0,00717	0,515	46,79	100,96	5,910	2,402	40,7	125,4
	2	3	172,0	"	0,515	44,58	100,76	5,897	3,125	52,9	171,4
	3	3	216,0	"	0,515	42,38	100,46	5,879	3,730	63,5	215,5
	4	3	255,3	"	0,515	40,18	100,96	5,910	4,179	70,7	254,1
	5	3	290,0	"	0,515	37,97	100,96	5,910	4,489	75,9	288,6
	6	3	320,3	"	0,518	35,77	100,36	5,908	4,669	79,0	319,7
	7	3	352,7	"	0,518	33,56	100,46	5,914	4,823	81,6	351,9
	8	3	384,3	"	0,518	31,36	100,86	5,937	4,911	82,7	382,7
	9	3	407,3	"	0,518	29,15	100,26	5,902	4,838	82,0	406,9
	10	3	439,7	"	0,520	26,95	100,86	5,960	4,828	81,0	437,9
	11	4	462,5	"	0,521	24,74	100,26	5,936	4,664	78,6	462,0
	12	3	486,0	"	0,515	22,54	100,26	5,867	4,463	76,1	485,5
	13	3	508,0	"	0,515	20,33	100,46	5,879	4,208	71,6	506,8
	14	3	530,0	"	0,517	18,19	100,26	5,885	3,923	66,7	529,4
	15	4	555,3	"	0,518	15,93	100,36	5,908	3,605	61,0	554,3
	16	3	574,0	"	0,520	13,72	100,86	5,960	3,209	53,8	571,6
	17	3	600,0	"	0,518	11,52	100,66	5,925	2,817	47,6	598,1
	18	3	624,0	"	0,518	9,31	100,16	5,896	2,368	40,2	623,5
	19	4	645,3	"	0,518	7,11	100,26	5,902	1,869	31,7	644,6
	20	5	625,4	"	0,519	9,31	100,36	5,919	2,377	40,2	624,3
	21	3	602,0	"	0,521	11,52	100,86	5,966	2,826	47,4	599,5
	22	3	576,7	"	0,518	13,72	100,66	5,925	3,224	54,4	574,9
	23	3	553,7	"	0,525	15,93	100,46	5,993	3,594	60,0	552,5
	24	3	527,7	"	0,522	18,19	100,26	5,947	3,912	65,7	527,2
	25	3	508,3	"	0,518	20,33	100,56	5,919	4,211	71,1	506,9
	26	3	486,3	"	0,518	22,54	100,46	5,913	4,466	75,5	485,2
	27	3	458,7	"	0,518	24,75	100,46	5,913	4,626	78,2	457,6
	28	3	435,0	"	0,517	26,95	100,36	5,890	4,776	81,1	434,5
	29	3	410,3	"	0,518	29,15	100,66	5,925	4,873	82,3	409,0
	30	4	382,0	"	0,518	31,36	100,26	5,902	4,881	82,7	381,6
	31	3	345,7	"	0,518	33,56	100,26	5,902	4,727	80,1	345,4
	32	4	318,0	"	0,518	35,77	100,16	5,896	4,635	78,6	317,8
	33	3	285,3	"	0,518	37,97	100,26	5,902	4,414	74,8	284,8
	34	3	253,0	"	0,518	40,18	100,36	5,907	4,142	70,1	252,6
	35	3	217,3	"	0,518	42,38	100,96	5,943	3,752	63,1	216,3
	36	2	174,5	"	0,518	44,58	100,56	5,919	3,170	53,5	174,0
	37	3	132,3	"	0,518	46,79	100,76	5,931	2,522	42,5	131,8
31st.	38	5	719,4	"	0,515	0	100,46	5,879	0	0	717,8

Temperature of water=76° F. Specific weight of water=62,5 lbs.

TABLE X.

Brake test of Böving impulse turbine.

Case II; needle stroke 0,308 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
29th. July 1920.	1	3	143,3	0,00857	0,604	53,41	100,66	6,909	3,119	45,2	142,8
	2	3	179,0	"	0,605	51,20	100,26	6,893	3,734	54,2	178,8
	3	3	215,0	"	0,603	48,99	100,06	6,856	4,292	62,6	215,0
	4	4	359,3	"	0,605	46,79	100,06	6,879	4,943	71,9	259,3
	5	3	285,7	"	0,606	44,58	100,36	6,911	5,190	75,1	285,2
	6	4	315,5	"	0,606	42,38	100,56	6,924	5,448	78,7	314,6
	7	8	345,8	"	0,606	40,18	100,96	6,952	5,661	81,4	344,2
	8	4	373,3	"	0,607	37,97	101,06	6,971	5,775	82,8	371,3
	9	3	392,3	"	0,603	35,77	99,96	6,850	5,718	83,5	392,4
	10	4	416,8	"	0,605	33,56	100,36	6,900	5,695	83,5	416,0
	11	3	440,7	"	0,605	31,36	100,36	6,900	5,631	81,6	439,9
	12	4	459,7	"	0,605	29,15	100,46	6,907	5,460	79,1	458,6
	13	3	478,0	"	0,604	26,95	100,26	6,892	5,248	76,3	477,5
	14	3	500,0	"	0,603	24,74	100,36	6,876	5,043	73,4	499,1
	15	3	520,0	"	0,605	22,54	100,36	6,900	4,776	69,2	519,1
	16	3	536,3	"	0,605	20,33	100,36	6,900	4,443	64,4	535,4
	17	3	559,3	"	0,605	18,19	100,36	6,900	4,146	60,1	558,3
	18	3	583,3	"	0,605	15,93	100,56	6,913	3,786	54,8	581,7
	19	4	603,0	"	0,605	13,72	101,26	6,962	3,371	48,4	599,0
	20	4	623,8	"	0,605	11,52	100,86	6,934	2,928	42,2	621,2
	21	3	640,7	"	0,605	9,31	100,96	6,941	2,431	35,0	637,7
	22	3	653,7	"	0,605	7,11	100,86	6,934	1,896	27,3	651,0
	23	3	640,7	"	0,605	9,31	100,96	6,941	2,431	35,0	637,7
	24	3	623,7	"	0,606	11,52	100,96	6,952	2,928	42,1	620,8
	25	3	600,7	"	0,606	13,72	100,36	6,910	3,359	48,6	599,6
	26	3	581,7	"	0,606	15,93	100,36	6,910	3,776	54,7	580,7
	27	3	561,7	"	0,607	18,19	100,36	6,916	4,164	60,2	560,7
	28	3	539,3	"	0,605	20,33	100,76	6,920	4,468	64,6	537,4
	29	3	519,3	"	0,607	22,54	100,86	6,957	4,769	68,6	517,1
	30	3	498,3	"	0,603	24,74	99,96	6,906	5,025	72,8	498,4
	31	3	479,7	"	0,603	26,95	99,96	6,849	5,267	76,9	479,8
	32	3	463,0	"	0,605	29,15	100,36	6,900	5,499	79,7	462,2
	33	3	440,0	"	0,605	31,36	100,26	6,893	5,622	81,6	439,0
	34	4	419,5	"	0,606	33,56	100,76	6,938	5,736	82,7	417,9
	35	3	395,7	"	0,605	35,77	100,76	6,927	5,768	83,3	394,2
	36	3	371,7	"	0,607	37,97	100,76	6,944	5,750	82,8	370,3
	37	3	346,7	"	0,605	40,18	100,56	6,913	5,676	82,1	345,8
	38	3	316,3	"	0,609	42,38	100,76	6,973	5,462	78,3	315,1
	39	3	283,7	"	0,609	44,58	100,36	6,943	5,154	74,2	283,2
	40	3	252,7	"	0,608	46,79	100,36	6,934	4,818	69,5	252,3
	41	3	221,0	"	0,609	48,99	100,36	6,945	4,411	63,5	220,6
	42	3	179,3	"	0,609	51,20	100,36	6,945	3,741	53,9	179,0
	43	3	140,3	"	0,608	53,41	100,36	6,934	3,053	44,0	140,0
31st.	44	4	717,5	"	0,601	0	100,26	6,847	0	0	716,8

Temperature of water=76° F. Specific weight of water=62,5 lbs.

TABLE XI.

Brake test of Böving impulse turbine.

Case II; needle stroke 1,000 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Dis-charge per second cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
31st. July 1920.	1	3	178,3	0,01017	0,685	57,82	99,77	7,765	4,201	54,1	178,5
	2	3	228,7	"	0,690	54,51	99,88	7,831	5,080	61,9	228,8
	3	3	274,7	"	0,690	51,20	99,88	7,831	5,731	73,2	274,9
	4	3	301,3	"	0,685	48,99	100,27	7,805	6,014	77,1	301,0
	5	4	329,6	"	0,690	46,79	99,98	7,839	6,283	80,2	329,6
	6	3	352,0	"	0,690	44,58	99,98	7,839	6,394	81,6	352,0
	7	4	371,4	"	0,690	42,38	99,78	7,823	6,413	82,0	371,8
	8	3	395,5	"	0,690	40,18	99,78	7,823	6,475	82,8	395,9
	9	4	415,5	"	0,690	37,97	99,83	7,831	6,428	82,1	415,7
	10	3	440,0	"	0,690	35,77	100,38	7,870	6,414	81,5	439,2
	11	3	455,3	"	0,690	33,56	99,98	7,839	6,225	79,4	455,3
	12	4	472,8	"	0,690	31,36	99,98	7,839	6,041	77,1	472,8
	13	3	485,0	"	0,690	29,15	100,18	7,855	5,761	73,3	484,6
	14	4	506,5	"	0,990	26,95	99,98	7,839	5,561	70,9	506,5
	15	4	534,3	"	0,690	23,64	99,98	7,839	5,147	65,7	534,3
	16	3	561,7	"	0,690	20,33	100,18	7,854	4,653	59,2	561,2
	17	3	589,0	"	0,620	17,03	99,98	7,839	4,060	51,8	585,0
	18	4	613,0	"	0,690	13,72	99,98	7,839	3,427	43,7	613,0
	19	4	640,5	"	0,690	10,42	99,88	7,831	2,719	34,7	640,9
	20	3	664,0	"	0,690	7,11	99,78	7,823	1,924	24,6	664,7
	21	3	640,0	"	0,690	10,42	100,08	7,846	2,717	34,6	639,8
	22	3	617,0	"	0,692	13,72	99,88	7,854	3,449	43,9	617,4
	23	4	587,5	"	0,692	17,03	100,08	7,869	4,077	51,8	587,2
	24	3	561,7	"	0,691	20,33	99,88	7,842	4,653	59,3	562,1
	25	3	529,7	"	0,690	23,64	99,88	7,831	5,102	65,2	530,0
	26	2	506,0	"	0,690	26,95	100,78	7,901	5,556	70,3	504,6
	27	3	488,0	"	0,690	29,15	99,78	7,823	5,796	74,1	488,6
	28	3	473,0	"	0,690	31,36	99,78	7,823	6,044	77,3	473,5
	29	3	453,7	"	0,685	33,56	99,87	7,774	6,204	79,8	454,0
	30	3	435,0	"	0,690	35,77	99,88	7,831	6,241	81,0	435,3
	31	3	411,7	"	0,690	37,97	99,78	7,823	6,370	81,4	412,1
	32	3	396,7	"	0,690	40,18	99,98	7,839	6,494	82,8	396,7
	33	3	371,0	"	0,690	42,38	99,78	7,823	6,406	81,9	371,4
	34	3	349,0	"	0,690	44,58	99,78	7,823	6,340	81,0	349,4
	35	4	324,5	"	0,690	46,79	99,88	7,831	6,186	79,0	324,7
	36	3	301,7	"	0,690	48,99	100,08	7,847	6,022	76,8	301,6
	37	3	271,7	"	0,690	51,20	99,78	7,823	5,668	72,6	272,0
	38	3	228,3	"	0,690	54,51	99,78	7,823	5,071	64,8	228,5
	39	3	179,0	"	0,690	57,82	99,98	7,839	4,217	53,8	179,0
	40	3	134,0	"	0,685	60,02	100,57	7,823	3,277	41,9	133,6
	41	4	719,8	"	0,685	0	100,27	7,805	0	0	719,1

Temperature of wat r=76° F. Specific weight of water=62,5 lbs.

TABLE XII.

Brake test of Böving impulse turbine.

Case III; needle stroke 0,183 in. Brake Arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second cu. ft.	Brake load lbs.	Total available head h ft.	Total available h. f.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
9th. July 1920.	1	3	52,0	0,00252	0,165	15,93	100,29	1,880	0,338	18,0	51,9
	2	3	73,0	"	0,167	14,82	100,29	1,903	0,441	23,2	72,9
	3	6	121,2	"	0,166	13,72	100,19	1,890	0,678	35,9	121,1
	4	3	204,5	"	0,166	12,62	99,89	1,885	1,052	55,8	204,6
	5	3	254,0	"	0,167	11,52	100,59	1,909	1,192	62,5	253,3
	6	4	301,3	"	0,167	10,42	100,09	1,899	1,279	67,4	301,2
	7	4	341,0	"	0,167	9,31	99,99	1,898	1,294	68,2	341,0
	8	4	390,3	"	0,167	8,21	100,29	1,903	1,306	68,6	389,8
	9	4	432,0	"	0,167	7,11	99,89	1,896	1,252	66,0	432,3
	10	3	472,7	"	0,168	6,01	100,19	1,912	1,157	60,5	472,2
	11	3	434,7	"	0,168	7,11	100,19	1,913	1,259	65,8	434,3
	12	3	489,0	"	0,168	8,21	99,59	1,901	1,302	68,5	387,9
	13	4	343,7	"	0,167	9,31	100,29	1,903	1,304	68,6	343,3
	14	4	300,0	"	0,168	10,42	99,99	1,909	1,274	66,7	300,0
	15	3	252,7	"	0,168	11,52	100,09	1,911	1,186	62,1	252,6
	16	4	208,3	"	0,168	12,62	100,39	1,917	1,048	54,7	203,4
	17	3	155,0	"	0,168	13,72	100,09	1,911	0,867	45,4	154,9
	18	3	78,0	"	0,168	14,82	99,79	1,905	0,471	24,7	78,1
	19	3	60,0	"	0,168	15,93	100,09	1,911	0,390	20,4	60,0
21st.	20	4	645,5	"	0,168	0	100,09	1,911	0	0	645,2

Temperature of water=69° F. Specific weight of water=62,5 lbs.

TABLE XIII.

Brake test of Böving impulse turbine.

Case III; needle stroke 0,400 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second q cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
19th. July 1920.	1	2	83,0	0,00482	0,334	31,36	99,98	3,795	1,061	28,0	83,0
	2	3	147,7	"	0,340	29,15	100,19	3,871	1,754	45,3	147,6
	3	3	196,7	"	0,342	26,95	100,19	3,893	2,160	55,5	196,5
	4	3	254,7	"	0,342	24,74	99,99	3,886	2,568	66,1	254,7
	5	3	303,7	"	0,343	22,54	99,99	3,897	2,789	71,6	303,8
	6	5	334,4	"	0,344	21,43	100,09	3,913	2,920	74,6	334,3
	7	4	357,3	"	0,344	20,33	100,09	3,913	2,960	75,7	357,2
	8	4	381,3	"	0,343	19,23	100,29	3,909	2,987	76,4	380,8
	9	4	404,3	"	0,342	18,19	100,09	3,890	2,994	77,0	404,2
	10	3	429,0	"	0,342	17,03	100,19	3,893	2,977	76,5	428,6
	11	3	445,3	"	0,342	15,93	99,89	3,882	2,890	74,5	445,6
	12	3	466,7	"	0,342	14,82	100,09	3,890	2,818	72,5	466,5
	13	4	487,0	"	0,342	13,72	100,29	3,897	2,722	69,9	486,3
	14	3	520,3	"	0,342	11,52	100,09	3,890	2,444	62,8	520,1
	15	3	552,3	"	0,342	9,31	100,09	3,890	2,096	53,9	552,1
	16	3	581,7	"	0,344	7,11	100,09	3,913	1,685	43,1	581,5
	17	5	549,0	"	0,344	9,31	100,09	3,913	2,083	53,2	548,8
	18	4	518,5	"	0,343	11,52	100,09	3,901	2,434	62,4	518,3
	19	4	483,8	"	0,344	13,72	100,09	3,913	2,704	69,1	483,6
	20	3	466,7	"	0,343	14,82	100,49	3,917	2,818	72,0	465,7
	21	4	446,5	"	0,343	15,93	100,19	3,905	2,898	74,2	446,1
	22	4	427,0	"	0,344	17,03	100,29	3,920	2,963	75,6	426,4
	23	3	407,7	"	0,344	18,19	100,19	3,916	3,022	77,2	407,3
	24	3	384,7	"	0,344	19,23	100,56	3,932	3,014	76,7	383,6
	25	4	361,5	"	0,344	20,33	100,09	3,913	2,994	76,5	361,4
	26	3	340,3	"	0,344	21,43	99,99	3,909	2,972	76,0	340,4
	27	4	313,3	"	0,344	22,54	100,09	3,913	2,878	73,6	313,2
	28	4	259,7	"	0,344	24,74	100,09	3,913	2,618	66,9	259,6
	29	3	208,3	"	0,344	26,95	100,39	3,924	2,287	58,3	208,2
	30	3	148,7	"	0,344	29,15	99,79	3,901	1,766	45,3	148,9
	31	2	88,5	"	0,344	31,36	100,39	3,924	1,131	28,8	88,3
21st.	32	5	695,4	"	0,342	0	100,29	3,897	0	0	694,4

Temperature of water = 71° F. Specific weight of water = 62,5 lbs.

TABLE XIV.

Brake test of Böving impulse turbine.

Case III; needle stroke 0,650 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second q cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
19th. July 1920.	1	3	90,0	0,00717	0,515	46,79	100,55	5,884	1,716	29,2	89,7
	2	3	143,0	"	0,520	44,58	100,55	5,941	2,598	43,7	142,6
	3	3	179,0	"	0,520	42,38	100,55	5,941	3,091	52,0	178,5
	4	4	219,3	"	0,518	40,18	100,25	5,901	3,590	60,8	219,1
	5	3	254,0	"	0,520	37,97	100,25	5,924	3,930	66,3	253,7
	6	3	293,3	"	0,520	35,77	100,35	5,930	4,275	72,1	292,8
	7	3	331,7	"	0,520	33,56	100,25	5,924	4,536	76,6	331,4
	8	3	364,0	"	0,520	31,36	100,35	5,930	4,651	78,4	363,4
	9	3	394,7	"	0,520	29,15	100,45	5,915	4,688	79,0	393,8
	10	3	426,7	"	0,520	26,95	100,65	5,947	4,686	78,8	425,4
	11	3	454,3	"	0,520	24,74	100,35	5,930	4,579	77,2	453,5
	12	4	477,8	"	0,520	22,54	100,25	5,924	4,388	74,1	477,4
	13	4	505,0	"	0,520	20,33	100,55	5,941	4,183	70,4	503,5
	14	3	526,7	"	0,520	18,19	100,55	5,941	3,904	65,7	525,1
	15	3	549,7	"	0,520	15,93	100,35	5,930	3,568	60,2	548,8
	16	3	577,0	"	0,520	13,72	101,05	5,971	3,226	54,0	574,0
	17	3	595,7	"	0,520	11,52	100,55	5,941	2,796	47,1	594,0
	18	3	616,3	"	0,520	9,31	100,25	5,924	2,339	39,5	615,7
	19	3	638,3	"	0,520	7,11	100,25	5,924	1,849	31,2	637,4
	20	3	616,0	"	0,520	9,31	100,45	5,935	2,338	39,4	614,6
	21	3	593,7	"	0,520	11,52	100,25	5,924	2,787	47,1	593,1
	22	3	570,3	"	0,520	13,72	100,55	5,941	3,188	53,7	568,6
	23	3	547,3	"	0,520	15,93	100,65	5,947	3,553	59,8	545,6
	24	3	523,3	"	0,521	18,19	100,25	5,935	3,879	65,4	522,8
	25	4	501,8	"	0,520	20,33	100,25	5,924	4,156	70,2	501,3
	26	3	480,0	"	0,520	22,54	100,25	5,924	4,408	74,4	479,5
	27	3	451,7	"	0,520	24,74	100,25	5,924	4,553	76,9	451,3
	28	3	422,7	"	0,520	26,95	100,35	5,930	4,642	78,3	422,0
	29	3	396,3	"	0,520	29,15	99,95	5,906	4,707	79,7	396,4
	30	3	366,3	"	0,518	31,36	100,35	5,907	4,681	79,2	365,6
	31	4	334,3	"	0,520	33,56	100,05	5,912	4,571	77,3	334,2
	32	3	297,3	"	0,520	35,77	100,25	5,924	4,333	73,2	297,0
	33	3	257,7	"	0,520	37,97	100,25	5,924	3,987	67,3	257,4
	34	3	224,0	"	0,520	40,18	100,45	5,935	3,668	61,8	223,5
	35	3	180,7	"	0,520	42,38	100,45	5,935	3,120	52,6	180,3
	36	1	141,0	"	0,517	44,58	100,75	5,919	2,561	43,3	140,5
	37	3	87,3	"	0,516	46,79	100,35	5,884	1,664	28,3	87,3
21st.	38	5	713,4	"	0,515	0	100,45	5,879	0	0	711,8

Temperature of water=71° F. Specific weight of water=62,5 lbs.

TABLE XV.

Brake test of Böving impulse turbine.

Case III; needle stroke 0.808 in. Brake arm 2.14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
20th. July 1920.	1	2	100.0	0.00857	0.603	53.41	100.36	6.876	2.176	31.7	99.8
	2	4	147.5	"	0.604	51.20	100.06	6.868	2.747	40.0	147.5
	3	4	182.0	"	0.605	48.99	100.06	6.879	3.633	52.8	182.0
	4	4	216.7	"	0.603	46.79	99.76	6.836	4.131	60.4	217.0
	5	3	247.3	"	0.603	44.58	100.16	6.863	4.492	65.5	247.1
	6	3	282.3	"	0.604	42.38	99.96	6.861	4.875	71.1	282.4
	7	3	318.3	"	0.603	40.18	100.26	6.870	5.211	75.9	318.0
	8	4	349.0	"	0.605	37.97	100.26	6.893	5.399	78.3	348.7
	9	4	377.0	"	0.606	35.77	100.56	6.925	5.494	79.3	376.0
	10	3	403.3	"	0.605	33.56	100.36	6.900	5.514	79.9	402.6
	11	3	426.3	"	0.605	31.36	100.26	6.893	5.447	79.0	425.9
	12	3	450.0	"	0.605	29.15	100.36	6.900	5.345	77.5	449.2
	13	3	473.7	"	0.605	26.95	100.56	6.914	5.202	75.2	472.4
	14	3	496.3	"	0.606	24.74	100.66	6.931	5.003	72.2	494.7
	15	3	513.0	"	0.606	22.54	100.56	6.924	4.711	68.0	511.6
	16	4	533.3	"	0.606	20.33	100.36	6.910	4.417	63.9	532.4
	17	3	554.7	"	0.606	18.19	100.66	6.931	4.112	59.3	553.0
	18	3	571.7	"	0.606	15.93	100.46	6.917	3.711	53.7	570.4
	19	4	590.0	"	0.606	13.72	100.36	6.910	3.298	47.7	588.9
	20	3	610.7	"	0.606	11.52	100.66	6.931	2.867	41.4	608.8
	21	3	631.0	"	0.605	9.31	100.36	6.900	2.394	34.7	629.9
	22	3	647.3	"	0.605	7.11	100.16	6.886	1.875	27.2	646.8
	23	3	631.3	"	0.605	9.31	100.46	6.906	2.395	34.7	629.9
	24	3	608.0	"	0.605	11.52	100.26	6.893	2.854	41.4	607.4
	25	4	591.0	"	0.605	13.72	100.36	6.900	3.304	47.9	589.9
	26	3	571.3	"	0.605	15.93	100.36	6.900	3.708	53.7	570.3
	27	4	553.5	"	0.605	18.19	100.36	6.900	4.103	59.5	552.6
	28	3	534.7	"	0.606	20.33	100.56	6.924	4.429	64.0	533.3
	29	3	517.7	"	0.606	22.54	100.96	6.952	4.755	68.4	515.3
	30	4	496.5	"	0.606	24.74	100.46	6.917	5.005	72.4	495.4
	31	4	474.0	"	0.606	26.95	100.36	6.910	5.205	75.3	473.2
	32	4	452.5	"	0.606	29.15	100.46	6.917	5.375	77.7	451.5
	33	3	428.7	"	0.606	31.36	99.96	6.883	5.478	79.6	428.8
	34	3	405.3	"	0.606	33.56	100.36	6.910	5.541	80.2	404.6
	35	3	382.7	"	0.606	35.77	100.76	6.938	5.577	80.4	381.2
	36	3	356.3	"	0.608	37.97	100.86	6.968	5.512	79.1	354.8
	37	3	323.3	"	0.606	40.18	100.86	6.945	5.293	76.2	321.9
	38	3	288.3	"	0.606	42.38	100.36	6.910	4.978	72.1	287.8
	39	3	249.3	"	0.606	44.58	100.16	6.897	4.529	65.7	249.1
	40	3	220.0	"	0.605	46.79	100.96	6.941	4.194	60.4	219.0
	41	3	183.7	"	0.605	48.99	100.36	6.900	3.667	53.1	183.4
	42	2	148.5	"	0.605	51.20	100.46	6.906	2.766	40.1	148.2
	43	2	106.0	"	0.606	53.41	100.86	6.945	2.507	33.2	105.6
21st.	44	6	716.8	"	0.605	0	100.56	6.913	0	0	714.8

Temperature of water = 71° F. Specific weight of water = 62.5 lbs.

TABLE XVI.

Brake test of Böving impulse turbine.

Case III; needle stroke 1,000 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
26th. July 1920.	1	3	160,7	0,01017	0,690	57,82	101,07	7,924	3,787	47,8	159,9
	2	3	195,7	"	0,690	54,51	100,77	7,900	4,347	55,0	195,0
	3	4	242,3	"	0,690	51,20	100,67	7,893	5,055	64,1	241,5
	4	3	291,3	"	0,688	47,90	100,37	7,847	5,686	72,5	290,8
	5	3	331,0	"	0,690	44,58	100,47	7,877	6,012	76,3	330,2
	6	3	360,0	"	0,692	42,38	100,47	7,900	6,282	79,5	359,2
	7	3	384,0	"	0,690	40,18	100,47	7,877	6,286	79,8	383,1
	8	3	404,7	"	0,690	37,97	100,47	7,877	6,261	79,5	403,8
	9	3	428,3	"	0,690	35,77	100,77	7,900	6,243	79,0	426,7
	10	3	451,7	"	0,690	33,56	100,57	7,885	6,177	78,3	450,4
	11	3	476,7	"	0,690	31,36	100,87	7,908	6,091	77,0	474,7
	12	4	486,5	"	0,692	29,15	100,47	7,900	5,778	73,1	485,4
	13	3	499,7	"	0,690	26,95	100,47	7,877	5,490	69,7	498,6
	14	3	529,7	"	0,690	23,64	100,27	7,861	5,103	64,9	529,1
	15	5	559,7	"	0,690	20,33	100,97	7,916	4,636	58,6	557,1
	16	3	584,3	"	0,692	17,03	100,57	7,907	4,055	51,3	582,6
	17	4	606,3	"	0,690	13,72	100,37	7,869	3,390	43,1	605,2
	18	3	637,0	"	0,690	10,42	100,47	7,877	2,705	34,3	635,5
	19	3	665,0	"	0,690	7,11	100,27	7,861	1,926	24,5	664,3
	20	3	638,8	"	0,690	10,42	100,17	7,854	2,714	34,6	638,3
	21	3	611,3	"	0,690	13,72	100,27	7,861	3,417	43,5	610,7
	22	3	582,7	"	0,690	17,03	100,17	7,854	4,044	51,5	582,2
	23	3	556,0	"	0,690	20,33	100,27	7,861	4,606	58,6	555,5
	24	3	528,0	"	0,690	23,64	100,17	7,854	5,086	64,8	527,6
	25	4	517,0	"	0,690	26,95	100,47	7,877	5,680	72,1	515,8
	26	3	483,0	"	0,690	29,15	100,37	7,869	5,737	72,9	482,2
	27	4	465,8	"	0,690	31,36	100,37	7,869	5,951	75,6	464,9
	28	4	448,0	"	0,690	33,56	100,47	7,877	6,126	77,8	447,0
	29	3	426,7	"	0,690	35,77	100,27	7,861	6,219	79,1	426,3
	30	3	401,7	"	0,688	37,97	100,67	7,823	6,214	79,4	401,6
	31	3	380,0	"	0,688	40,18	100,46	7,854	6,221	79,2	379,2
	32	3	358,0	"	0,688	42,38	100,26	7,838	6,247	79,7	357,7
	33	3	339,0	"	0,688	44,58	100,37	7,847	6,158	78,5	338,4
	34	3	283,3	"	0,690	47,90	101,07	7,924	5,568	70,3	283,8
	35	4	239,3	"	0,690	51,20	100,77	7,900	4,992	63,2	238,4
	36	3	183,7	"	0,691	54,51	100,47	7,888	4,124	52,3	183,3
	37	2	159,5	"	0,690	57,82	100,77	7,900	3,758	47,6	158,9
21st.	38	4	718,0	"	0,688	0	100,47	7,855	0	0	716,4

Temperature of water=71° F. Specific weight of water=62,5 lbs.

TABLE XVII.

Brake test of Böving impulse turbine.

Case IV; needle stroke 0,188 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second cu. ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
16th. June 1920.	1	3	46,5	0,00252	0,167	15,93	100,79	1,913	0,302	15,8	46,3
	2	5	77,2	"	0,167	14,82	100,19	1,901	0,466	24,5	77,1
	3	6	116,0	"	0,167	13,72	100,19	1,901	0,649	34,1	115,9
	4	3	196,0	"	0,168	12,62	100,09	1,901	1,008	52,8	195,9
	5	5	242,2	"	0,167	11,52	99,89	1,896	1,137	60,0	242,3
	6	4	297,5	"	0,168	10,42	100,49	1,918	1,263	65,9	296,8
	7	4	344,5	"	0,168	9,31	99,99	1,909	1,307	68,5	344,5
	8	3	375,7	"	0,167	8,21	99,59	1,890	1,257	66,5	376,5
	9	3	420,3	"	0,168	7,11	99,89	1,907	1,218	63,9	420,6
	10	3	473,3	"	0,168	6,01	100,19	1,913	1,159	60,6	472,8
	11	3	424,3	"	0,168	7,11	99,99	1,909	1,229	64,4	424,3
	12	5	387,0	"	0,168	8,21	100,29	1,915	1,295	67,6	386,5
	13	4	342,3	"	0,168	9,31	100,19	1,913	1,299	67,9	342,0
	14	3	300,7	"	0,169	10,42	100,69	1,934	1,277	66,0	299,7
	15	4	246,3	"	0,169	11,52	100,19	1,924	1,156	60,1	246,1
	16	3	200,0	"	0,169	12,62	100,69	1,934	1,028	53,2	199,3
	17	3	145,0	"	0,169	13,72	99,69	1,915	0,811	42,4	145,2
	18	4	85,5	"	0,168	14,82	100,09	1,910	0,516	27,0	85,5
	19	2	33,5	"	0,167	15,93	100,59	1,909	0,217	11,4	33,4
23rd.	20	4	642,5	"	0,168	0	100,79	1,913	0	0	640,0

Temperature of water = 67° F. Specific weight of water = 62,5 lbs.

TABLE XVIII.

Brake test of Böving impulse turbine.

Case IV; needle stroke 0,400 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second cubic ft.	Brake load lbs.	Total available head ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
16th. June 1920.	1	2	64,0	0,00482	0,338	31,36	100,19	3,848	0,818	21,3	63,9
	2	3	138,0	"	0,338	29,15	100,29	3,852	1,639	42,5	137,8
	3	3	188,3	"	0,338	26,95	100,99	3,879	2,668	53,3	187,4
	4	5	243,8	"	0,338	24,74	100,69	3,867	2,458	63,6	243,0
	5	3	297,3	"	0,338	22,54	100,19	3,848	2,730	71,0	297,0
	6	3	319,7	"	0,339	21,43	100,29	3,864	2,792	72,3	319,2
	7	3	339,7	"	0,339	20,33	100,49	3,871	2,814	72,7	338,9
	8	3	358,7	"	0,338	19,23	100,09	3,844	2,810	73,1	358,6
	9	3	397,0	"	0,340	18,19	100,69	3,890	2,943	75,6	395,7
	10	4	410,0	"	0,342	17,03	99,89	3,882	2,845	73,3	410,3
	11	3	436,0	"	0,339	15,93	100,29	3,864	2,830	73,2	435,4
	12	3	454,7	"	0,342	14,82	100,09	3,890	2,746	70,6	454,5
	13	3	477,3	"	0,338	13,72	100,29	3,852	2,668	69,3	476,6
	14	5	514,8	"	0,344	11,52	100,49	3,928	2,417	61,5	513,6
	15	3	547,3	"	0,343	9,31	100,39	3,913	2,077	53,1	546,3
	16	4	579,3	"	0,338	7,11	100,79	3,871	1,678	43,4	577,0
	17	3	545,3	"	0,342	9,31	100,39	3,901	2,069	53,0	544,3
	18	4	515,5	"	0,342	11,52	100,59	3,909	2,420	61,9	514,0
	19	4	475,0	"	0,338	13,72	100,59	3,864	2,656	68,7	473,7
	20	3	454,3	"	0,339	14,82	100,89	3,887	2,743	70,6	452,3
	21	5	432,4	"	0,338	15,93	100,09	3,844	2,807	73,0	432,3
	22	3	416,3	"	0,342	17,03	100,69	3,913	2,889	73,8	414,9
	23	5	389,0	"	0,338	18,19	100,49	3,860	2,883	74,7	388,1
	24	3	369,0	"	0,338	19,23	100,29	3,852	2,891	75,1	368,5
	25	5	343,2	"	0,341	20,33	100,99	3,913	2,843	72,7	341,6
	26	3	315,7	"	0,339	21,43	100,49	3,871	2,757	71,2	314,9
	27	3	292,0	"	0,338	22,54	100,69	3,867	2,682	69,4	291,0
	28	3	244,5	"	0,342	24,74	100,59	3,909	2,465	63,4	243,8
	29	4	188,0	"	0,340	26,95	100,09	3,367	2,064	53,4	187,9
	30	3	129,7	"	0,341	29,15	100,39	3,890	1,541	39,6	129,5
	31	3	59,0	"	0,340	31,36	100,29	3,875	0,754	19,5	58,9
23rd.	32	3	691,0	"	0,340	0	100,39	3,879	0	0	689,0

Temperature of water = 67° F. Specific weight of water = 62,5 lbs.

TABLE XIX.

Brake test of Böving impulse turbine.

Case IV; needle stroke 0,650 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Dis-charge per second a cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
17th. June 1920.	1	2	58,5	0,00717	0,510	46,79	100,75	5,838	1,115	19,1	58,3
	2	2	92,0	"	0,512	44,58	100,65	5,856	1,671	28,5	91,7
	3	3	145,0	"	0,510	42,38	100,35	5,816	2,504	43,1	144,7
	4	3	183,0	"	0,510	40,18	100,15	5,804	2,996	51,6	182,9
	5	3	222,0	"	0,510	37,97	100,25	5,810	3,434	59,1	221,7
	6	5	262,2	"	0,510	35,77	100,25	5,810	3,822	65,8	261,9
	7	4	301,7	"	0,512	33,56	100,55	5,850	4,125	70,5	300,9
	8	3	336,3	"	0,513	31,36	100,95	5,885	4,298	73,0	324,7
	9	3	372,3	"	0,511	29,15	100,45	5,833	4,421	75,8	371,5
	10	3	406,0	"	0,511	26,95	100,45	5,833	4,458	76,4	405,2
	11	3	438,3	"	0,513	24,74	100,95	5,885	4,419	75,1	436,2
	12	4	463,7	"	0,513	22,54	100,65	5,867	4,259	72,6	462,2
	13	5	488,6	"	0,512	20,33	100,35	5,838	4,048	69,3	487,8
	14	3	512,3	"	0,511	18,19	100,15	5,815	3,797	65,3	511,9
	15	3	540,0	"	0,511	15,93	100,45	5,833	3,509	60,2	538,9
	16	3	564,7	"	0,511	13,72	100,75	5,850	3,157	54,0	562,6
	17	3	586,3	"	0,513	11,52	100,95	5,885	2,752	46,8	583,5
	18	3	610,0	"	0,514	9,31	100,95	5,897	2,315	39,3	607,1
	19	3	628,0	"	0,512	7,11	100,15	5,827	1,819	31,2	627,5
	20	3	606,3	"	0,511	9,31	100,45	5,833	2,301	39,5	605,0
	21	4	583,3	"	0,512	11,52	100,55	5,850	2,738	46,8	581,7
	22	4	554,5	"	0,511	13,72	99,75	5,792	3,100	52,5	555,2
	23	4	535,0	"	0,511	15,93	100,45	5,833	3,476	59,6	533,8
	24	4	514,5	"	0,512	18,19	101,05	5,879	3,814	64,7	511,8
	25	3	489,3	"	0,514	20,33	100,85	5,891	4,053	68,8	487,3
	26	3	457,7	"	0,512	22,54	99,85	5,809	4,203	72,4	458,0
	27	3	434,3	"	0,513	24,74	100,45	5,856	4,378	74,8	433,4
	28	3	405,3	"	0,514	26,95	100,85	5,891	4,450	75,5	403,6
	29	4	372,8	"	0,514	29,15	100,95	5,897	4,427	75,1	371,0
	30	3	340,0	"	0,515	31,36	101,15	5,919	4,345	73,4	338,1
	31	4	303,5	"	0,514	33,56	100,55	5,874	4,150	70,7	302,7
	32	3	266,0	"	0,513	35,77	100,85	5,879	3,877	65,9	264,9
	33	3	225,3	"	0,511	37,97	100,35	5,827	3,485	59,8	224,9
	34	3	180,7	"	0,511	40,18	100,05	5,810	2,958	50,9	180,7
	35	3	151,0	"	0,512	42,38	101,05	5,879	2,608	44,4	150,2
	36	4	97,3	"	0,511	44,58	100,45	5,833	1,767	30,3	97,1
	37	2	64,0	"	0,512	46,79	100,85	5,867	1,220	20,8	63,7
23rd.	38	3	712,3	"	0,512	0	100,35	5,838	0	0	711,0

Temperature of water=67° F. Specific weight of water=62,5 lbs.

TABLE XX.

Brake test of Böving impulse turbine.

Case IV; needle stroke 0,808 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
21st. Jun ^o 1920.	1	1	72,0	0,00857	0,601	53,41	101,45	6,929	1,567	22,6	71,5
	2	3	112,7	"	0,603	51,20	101,06	6,924	2,351	34,0	112,1
	3	3	153,0	"	0,603	48,99	100,56	6,980	3,054	44,3	152,6
	4	3	182,3	"	0,601	46,79	100,05	6,833	3,475	50,9	182,3
	5	3	221,0	"	0,601	44,48	100,15	6,861	4,014	58,5	220,5
	6	3	252,0	"	0,603	42,38	100,36	6,876	4,351	63,3	251,6
	7	3	283,7	"	0,600	40,18	100,05	6,821	4,644	68,1	283,7
	8	3	316,0	"	0,601	37,97	100,05	6,823	4,889	71,6	315,9
	9	3	344,2	"	0,598	35,77	99,95	6,793	5,018	73,9	344,4
	10	3	375,3	"	0,600	33,56	100,05	6,821	5,132	75,2	375,2
	11	3	403,0	"	0,601	31,36	99,95	6,826	5,149	75,4	403,1
	12	5	430,8	"	0,600	29,15	100,05	6,821	5,117	75,0	430,7
	13	3	454,7	"	0,600	26,95	100,05	6,821	4,993	73,2	454,6
	14	3	480,0	"	0,603	24,74	100,36	6,876	4,839	70,4	479,1
	15	3	503,3	"	0,603	22,54	100,46	6,883	4,622	67,2	502,2
	16	5	522,8	"	0,601	20,33	100,05	6,833	4,331	63,4	522,7
	17	3	541,0	"	0,600	18,19	100,15	6,823	4,010	58,7	540,6
	18	3	561,3	"	0,601	15,93	100,05	6,823	3,636	53,2	561,2
	19	3	579,0	"	0,601	13,72	99,85	6,820	3,237	47,5	579,4
	20	3	601,7	"	0,598	11,52	99,95	6,793	2,824	41,6	601,8
	21	3	622,3	"	0,603	9,31	100,16	6,863	2,360	34,4	621,8
	22	3	646,0	"	0,603	7,11	100,66	6,897	1,871	27,1	643,9
	23	3	622,7	"	0,603	9,31	100,36	6,876	2,362	34,4	621,6
	24	3	602,3	"	0,603	11,52	99,96	6,849	2,827	41,3	602,5
	25	3	583,0	"	0,603	13,72	99,96	6,849	3,259	47,6	583,2
	26	3	564,7	"	0,603	15,93	100,06	6,856	3,658	53,4	564,6
	27	3	544,3	"	0,603	18,19	100,56	6,890	4,034	58,6	542,8
	28	3	523,7	"	0,601	20,33	100,75	6,881	4,338	63,0	521,7
	29	3	503,0	"	0,603	22,54	100,26	6,869	4,620	67,3	502,4
	30	3	480,3	"	0,603	24,74	100,36	6,876	4,842	70,4	479,5
	31	3	456,0	"	0,603	26,95	100,36	6,876	5,007	72,8	455,2
	32	3	435,3	"	0,603	29,15	100,66	6,897	5,171	75,0	433,6
	33	3	406,0	"	0,603	31,36	100,26	6,869	5,188	75,5	405,6
	34	3	379,7	"	0,603	33,56	100,46	6,883	5,191	75,4	378,8
	35	3	359,0	"	0,603	35,77	100,26	6,869	5,101	74,3	349,6
	36	4	315,2	"	0,603	37,97	99,96	6,849	4,878	71,2	315,4
	37	4	286,3	"	0,600	40,18	99,85	6,803	4,637	68,9	286,5
	38	3	254,3	"	0,603	42,38	100,46	6,883	4,391	63,8	253,7
	39	3	216,7	"	0,601	44,58	99,65	6,806	3,936	57,8	217,1
	40	2	182,0	"	0,600	46,79	99,95	6,815	3,470	50,9	182,0
	41	3	148,0	"	0,603	48,99	99,96	6,849	2,954	43,1	148,0
	42	4	101,7	"	0,601	51,20	100,05	6,833	2,120	31,0	101,7
	43	2	83,5	"	0,601	53,41	99,95	6,826	1,817	26,6	83,5
23rd.	44	3	716,0	"	0,600	0	100,15	6,828	0	0	715,4

Temperature of water=67° F. Specific weight of water=62,5 lbs.

TABLE XXI.

Brake test of Böving impulse turbine.

Case IV; needle stroke 1,000 in. Brake arm 2,14 ft.

Date.	No.	Duration of test min.	R.P.M.	Area of nozzle opening sq. ft.	Discharge per second q cubic ft.	Brake load lbs.	Total available head h ft.	Total available H. P.	B.H.P.	Gross eff. %	R.P.M. at 100 ft. head.
22nd. June 1920.	1	3	59,0	0,01017	0,682	57,82	100,26	7,769	2,238	28,8	94,9
	2	3	159,3	"	0,682	54,51	100,36	7,777	3,538	45,5	159,0
	3	3	199,3	"	0,683	51,20	99,97	7,758	4,158	53,6	199,4
	4	3	252,3	"	0,683	47,90	100,37	7,789	4,924	63,2	251,9
	5	3	293,3	"	0,683	44,58	100,47	7,797	5,328	68,3	292,6
	6	3	320,7	"	0,683	42,38	100,27	7,781	5,538	71,2	320,3
	7	3	350,0	"	0,683	40,18	100,37	7,789	5,730	73,6	349,4
	8	3	380,0	"	0,683	37,97	100,57	7,804	5,879	75,3	379,0
	9	3	402,3	"	0,685	35,77	100,07	7,788	5,864	75,3	402,2
	10	3	423,3	"	0,683	33,56	100,07	7,766	5,789	74,5	423,2
	11	3	446,7	"	0,683	31,36	100,07	7,766	5,708	73,5	446,5
	12	3	471,7	"	0,685	29,15	100,57	7,827	5,603	71,6	470,3
	13	3	490,3	"	0,684	26,95	100,07	7,777	5,284	69,2	490,2
	14	4	523,3	"	0,685	23,64	100,97	7,858	5,032	64,0	519,8
	15	3	546,3	"	0,685	20,33	100,37	7,812	4,526	57,9	545,4
	16	3	573,3	"	0,685	17,03	100,27	7,804	3,978	51,0	572,6
	17	3	599,0	"	0,685	13,72	100,57	7,827	3,348	42,8	597,4
	18	3	626,7	"	0,686	10,42	100,07	7,800	2,661	34,1	626,4
	19	4	653,3	"	0,683	7,11	100,27	7,781	1,892	24,3	652,4
	20	3	626,3	"	0,685	10,42	100,17	7,797	2,659	34,1	625,8
	21	4	599,0	"	0,684	13,72	100,27	7,793	3,348	43,0	598,2
	22	3	575,3	"	0,685	17,03	100,47	7,820	3,902	51,1	574,0
	23	4	546,0	"	0,685	20,33	100,77	7,843	4,523	57,7	544,0
	24	3	519,3	"	0,685	23,64	100,57	7,827	5,003	63,9	517,9
	25	3	493,0	"	0,685	26,95	101,07	7,867	5,413	68,8	491,5
	26	3	463,0	"	0,683	29,15	99,97	7,758	5,524	71,2	463,1
	27	4	447,3	"	0,685	31,36	100,37	7,812	5,715	73,2	446,5
	28	3	425,0	"	0,685	33,56	100,27	7,804	5,812	74,5	424,5
	29	3	403,0	"	0,685	35,77	101,07	7,867	5,903	75,0	402,9
	30	3	380,3	"	0,686	37,97	100,97	7,870	5,884	74,8	378,5
	31	3	350,7	"	0,685	40,18	100,67	7,835	5,742	73,3	349,6
	32	3	321,0	"	0,684	42,38	100,17	7,785	5,543	71,2	320,7
	33	3	293,0	"	0,685	44,58	100,46	7,820	5,321	68,1	292,3
	34	3	251,3	"	0,685	47,90	100,87	7,851	4,904	62,5	250,2
	35	3	207,0	"	0,685	51,20	100,87	7,851	4,318	55,0	206,1
	36	3	164,3	"	0,686	54,51	101,37	7,902	3,649	46,2	163,2
	37	3	109,7	"	0,685	57,82	101,17	7,874	2,584	32,8	109,1
23rd.	38	3	718,0	"	0,681	0	100,46	7,774	0	0	716,4

Temperature of water=68° F. Specific weight of water=62.5 lbs.

TABLE XXII.

Maximum efficiency for various nozzle openings and discharges.

Gate opening.	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	1
Needle stroke in inches.	0,188	0,400	0,650	0 808	1,000
Discharge in cubic feet per second. .	0,168	0,343	0,520	0,608	0,687
Maximum efficiency in %.					
Case : I	71,0	79,2	79,5	81,6	81,5
Case : II	70,7	79,4	82,5	83,2	82,8
Case : III	68,8	77,0	79,5	80,1	80,4
Case : IV	68,5	75,0	76,0	75,5	75,2

Summary.

It will be seen, from Fig. 11—15 and Table XXII, that the case II—when the central ridges stand at right angles to the jet axis at the mean position between the beginning and end of full jet—has the best efficiencies for all gates except for $\frac{1}{4}$ gate at which cases I and II have practically equal efficiency. At $\frac{7}{8}$ gate—which corresponds nearly to the normal load of the wheel—case II has an efficiency of 83,2 per cent. exactly 10 per cent. greater than that in case IV when the relative entrance velocity of the jet is at right angles to the ridge at the same mean position of the bucket. In the order of efficiencies, case I is next to case II, then comes case III and lastly case IV. These variations of maximum efficiencies in the four cases may be explained shortly as follows: when we remember that the buckets under consideration are so shaped that the normal cross sections of the double ellipsoidal deflecting surface of the bucket are all at right angles to the plane containing the central ridge and the exit edges (called the bucket plane hereafter), it is evident that the impulsive action of the jet will be most efficient if the buckets could move parallel to themselves standing always at right angles to the jet, in which case the absolute deflection of

the mid particles of the jet shall always take place on the axial plane through the jet parallel to the wheel axis. Actually, the bucket's inclination against the jet axis continually changes and the buckets so disposed as to make them move in a way nearest to the ideal motion above mentioned, at least during the most important part of the range of jet action will have the highest efficiency. From these considerations, it may naturally be expected that case II will show the highest efficiency of the four cases experimented upon. The deviation of the bucket plane from the normal to the jet axis in the four cases are shown in Fig. 16 in which it will be observed that the deviation is least in case II. The maximum deviation in this case is $\pm 8^{\circ} 5'$ while it is $+22^{\circ} 14'$ in case IV. When the deviation is excessive as in the latter case, that is to say, when the bucket plane is much inclined toward the impinging jet, the deflected stream is forced downward considerably from the axial plane of the jet parallel to the wheel axis, which phenomenon was clearly observable during the tests through glass holes on the cover of the wheel; as a consequence of this, the torque exerted by the impinging jet is much reduced.

If the central ridge be not parallel to the plane containing the exit edges as in some bucket, it seems advisable so to fix the buckets that the jet shall be at right angles to the major axial plane—if it may be so called—of the double ellipsoid at the mean position of bucket between the beginning and end of the full jet.

To end, I desire to express my acknowledgement to Mr. Tokunaga, assistant in the Laboratory and all other fellow-workers for their tedious work in the tests, and calculations of the results.

Fukuoka, December, 1920.

Fig. 6. Relation between efficiency and speed
at constant head 100 ft.
Needle stroke = 0.188 in

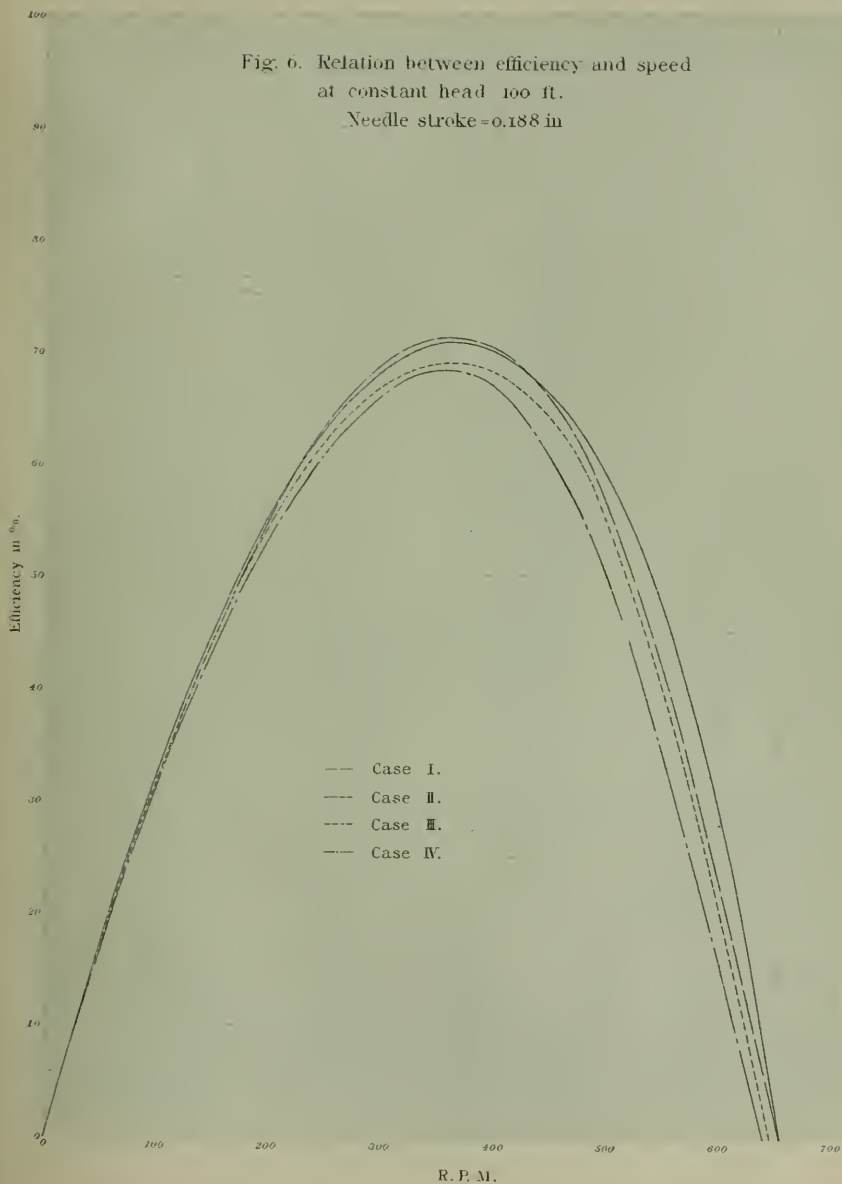
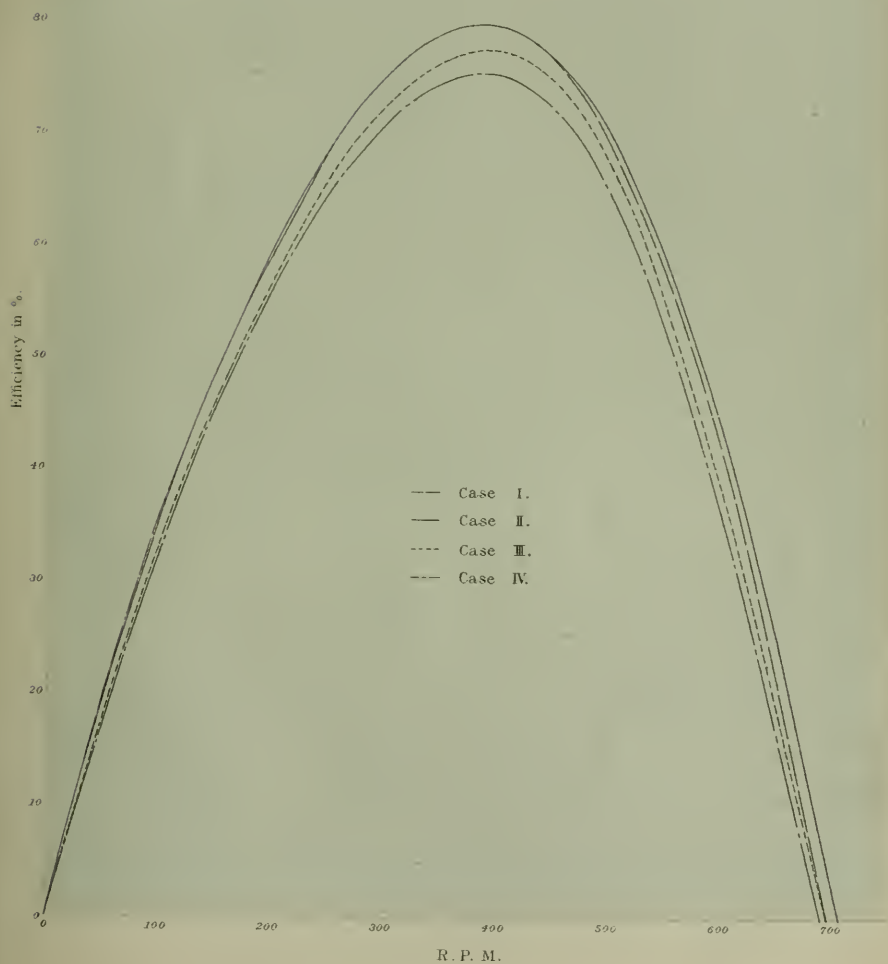
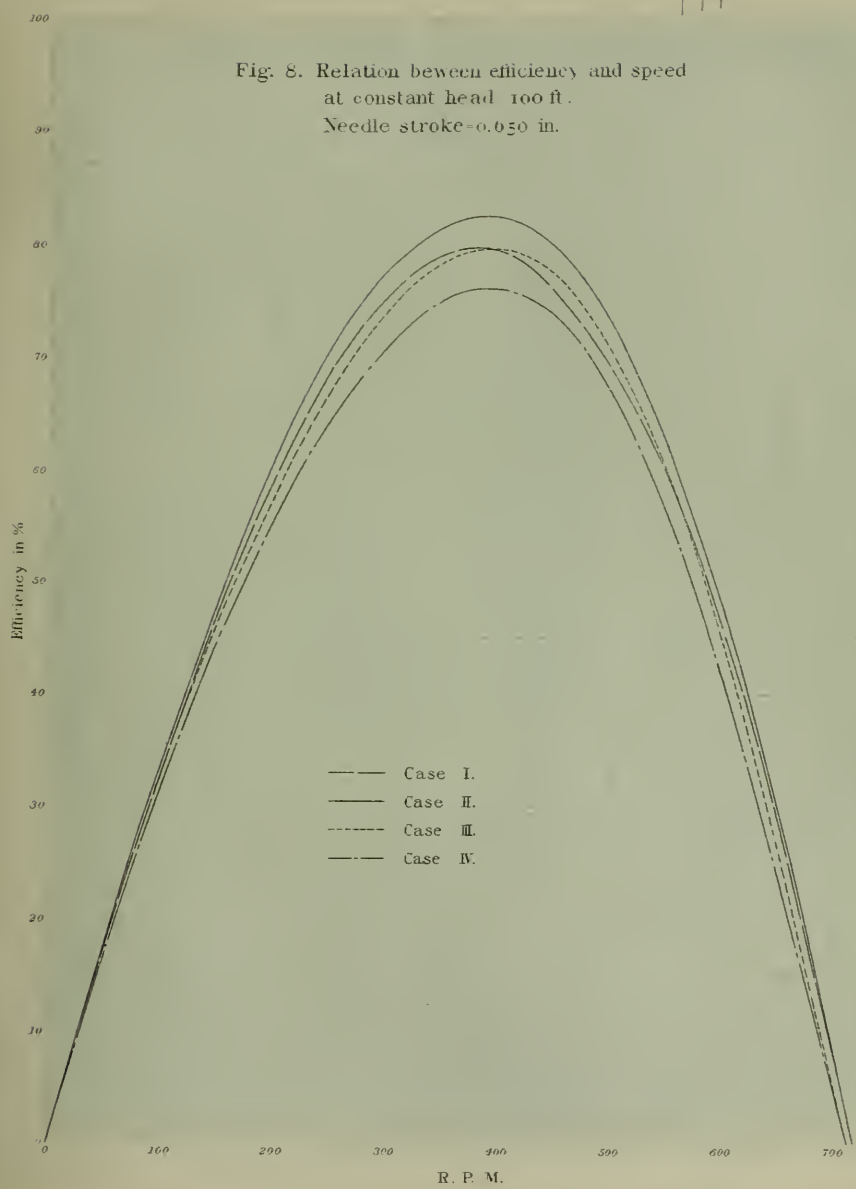


Fig. 7. Relation between efficiency and speed
at constant head 100 ft.
Needle stroke = 0.400 in.



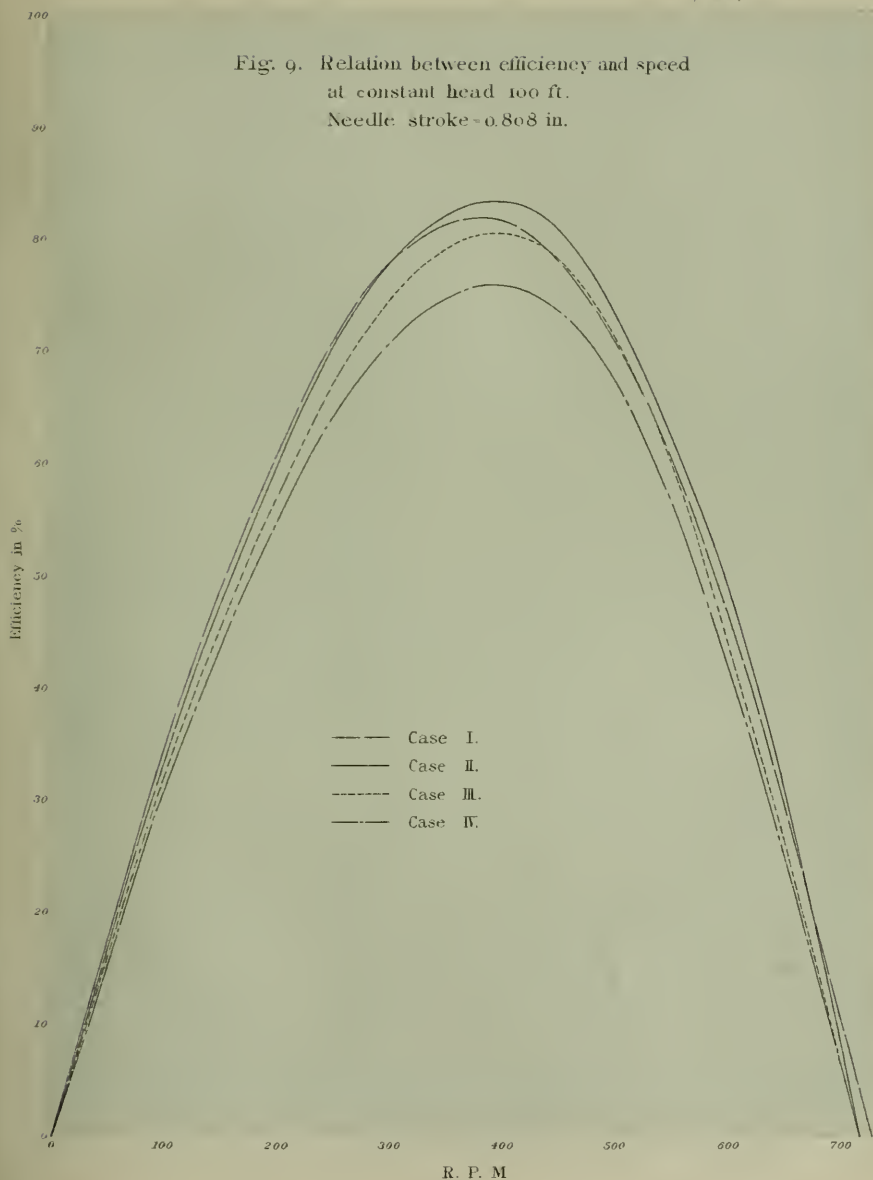
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Fig. 8. Relation between efficiency and speed
at constant head 100 ft.
Needle stroke=0.050 in.



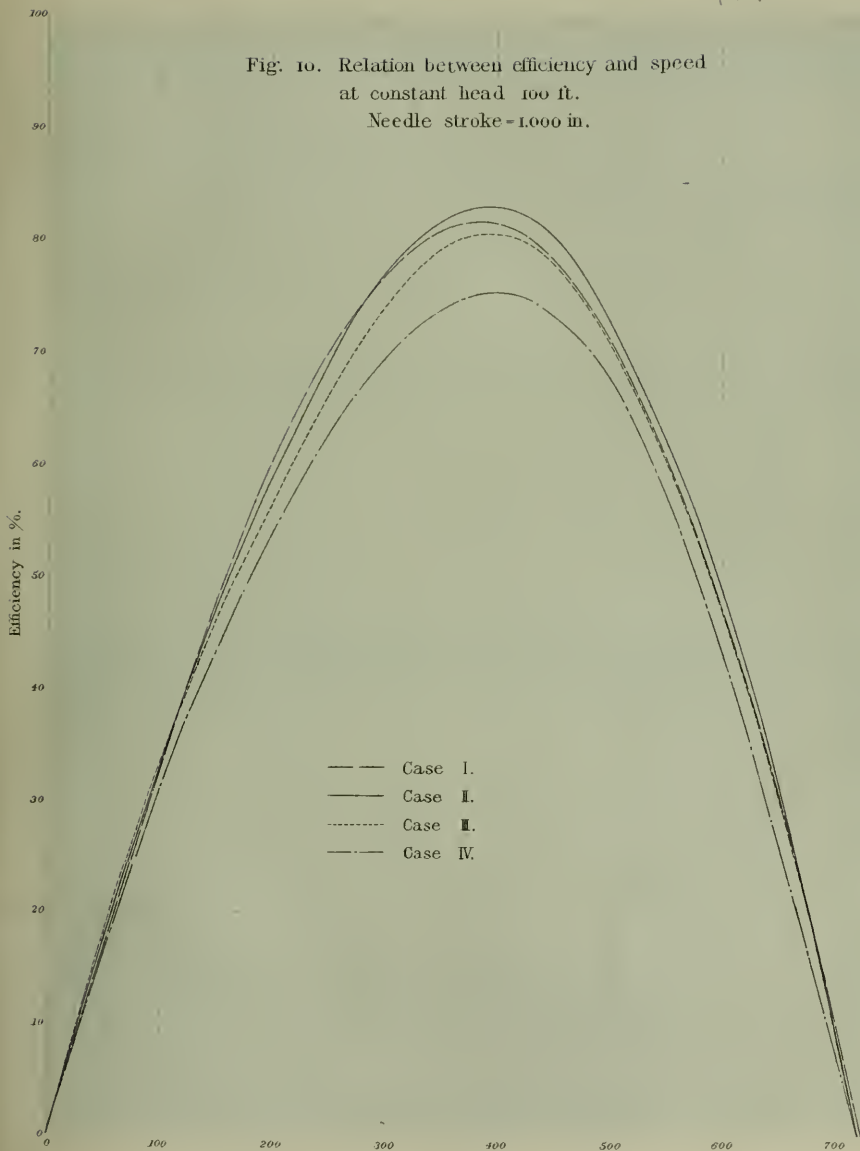
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Fig. 9. Relation between efficiency and speed
at constant head 100 ft.
Needle stroke = 0.808 in.



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Fig. 10. Relation between efficiency and speed
at constant head 100 ft.
Needle stroke = 1.000 in.



R. P. N.

Fig. 11. Characteristic Curve for Case I.

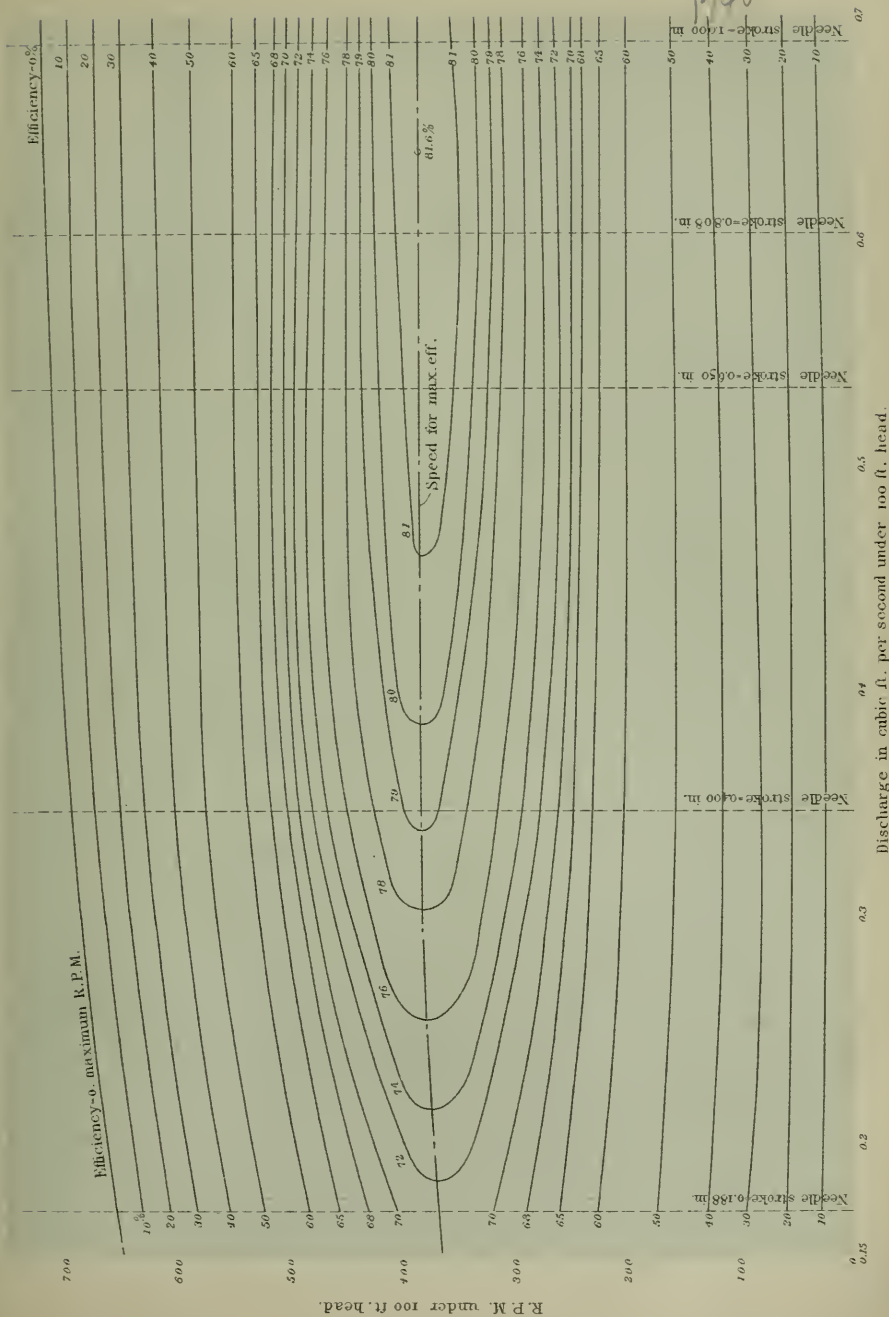


Fig. 12. Characteristic Curve for Case II.

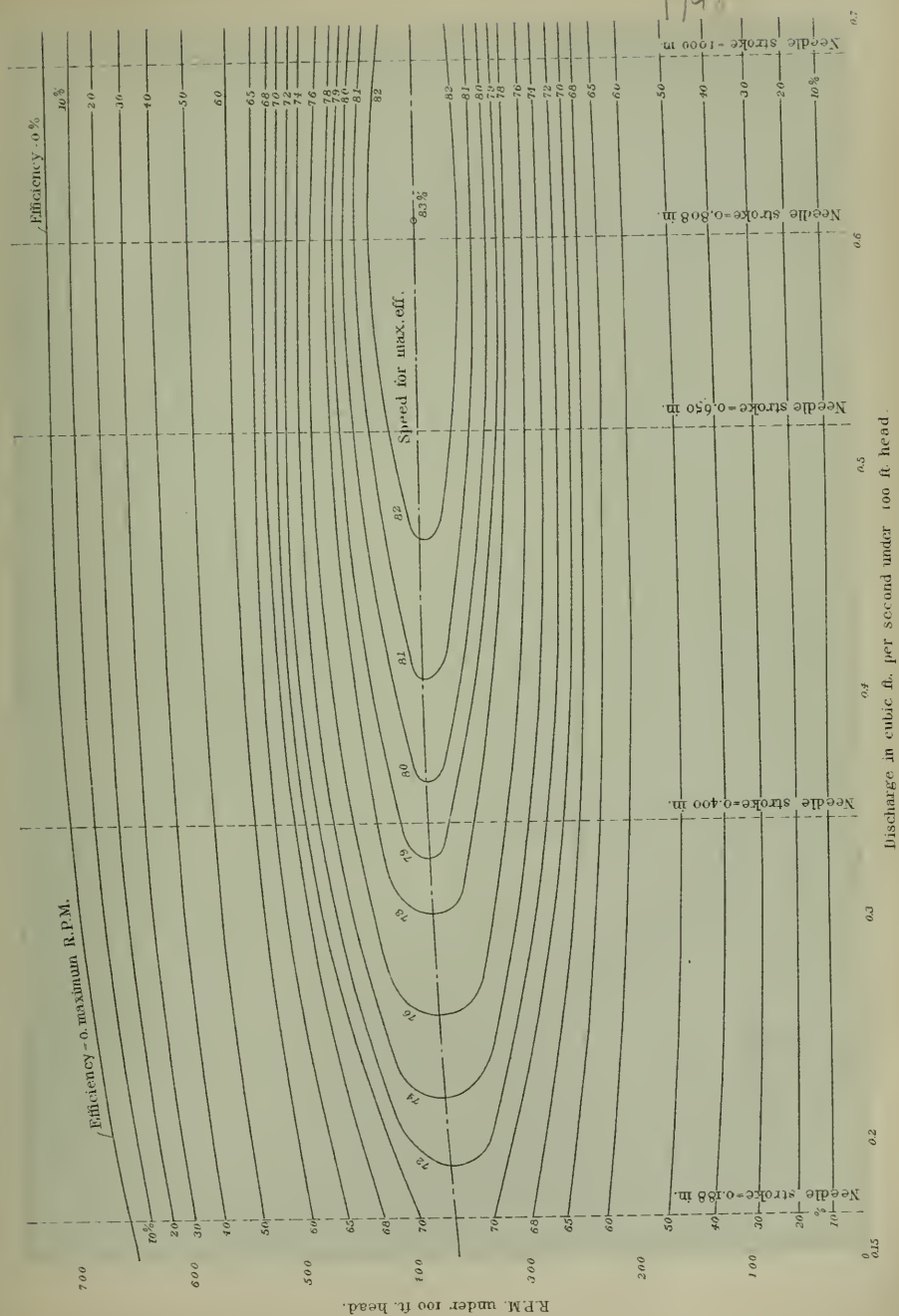
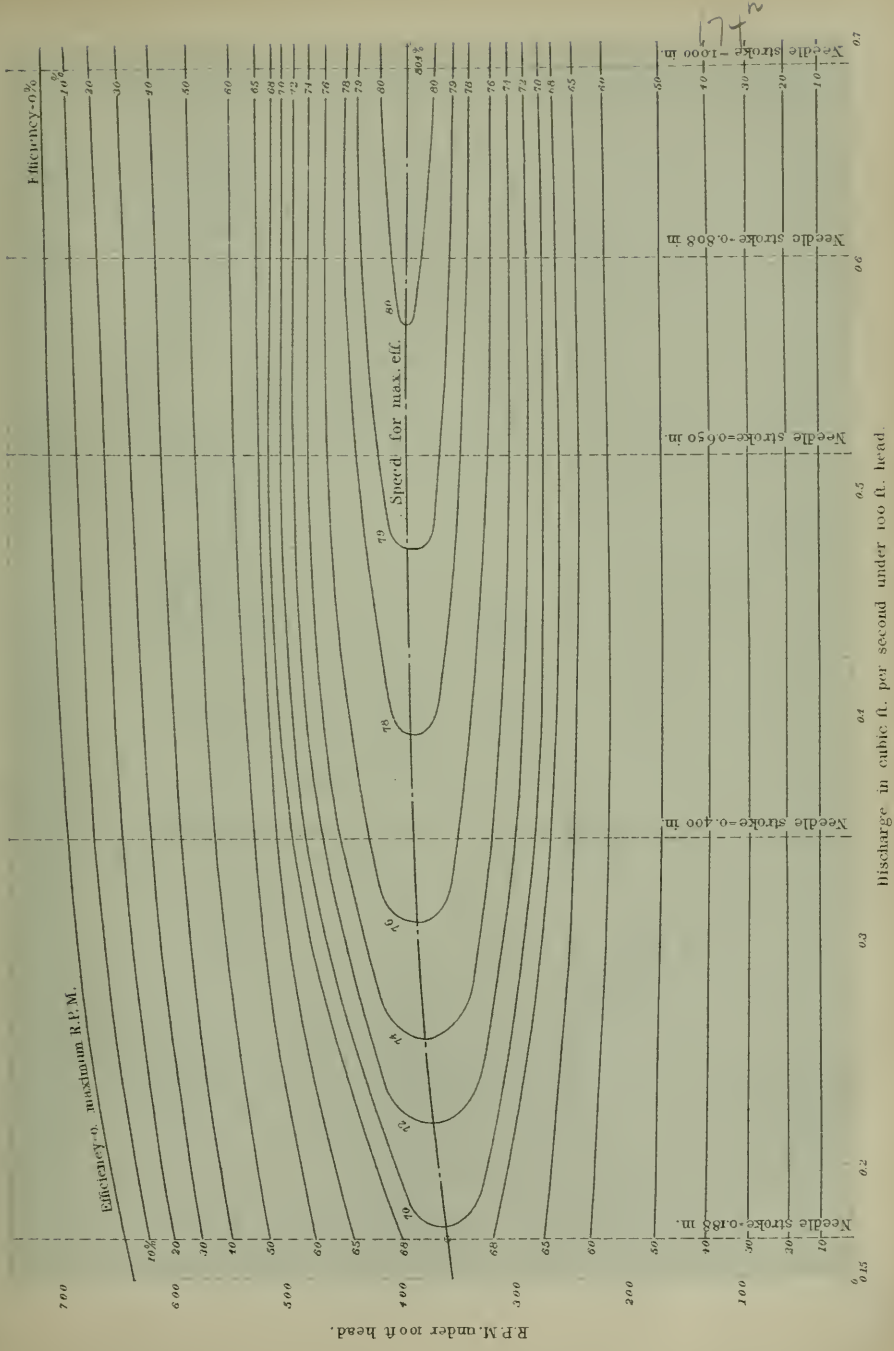


Fig. 13. Characteristic Curve for Case III.



R.P.M. under 100 ft head.

Fig. 14. Characteristic Curve for Case A.

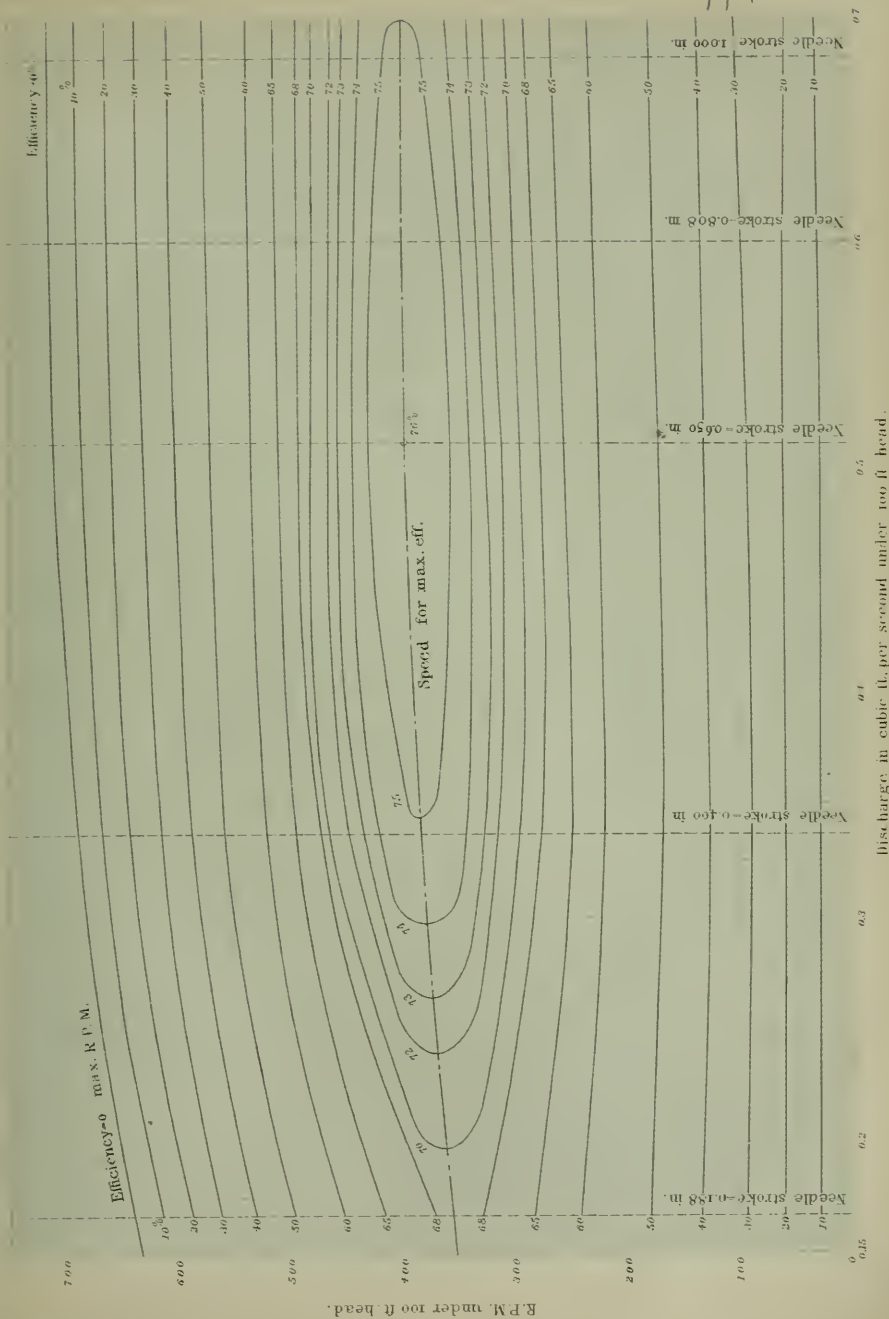


Fig. 15. Curves for Comparison of Maximum Efficiencies in Four Cases.

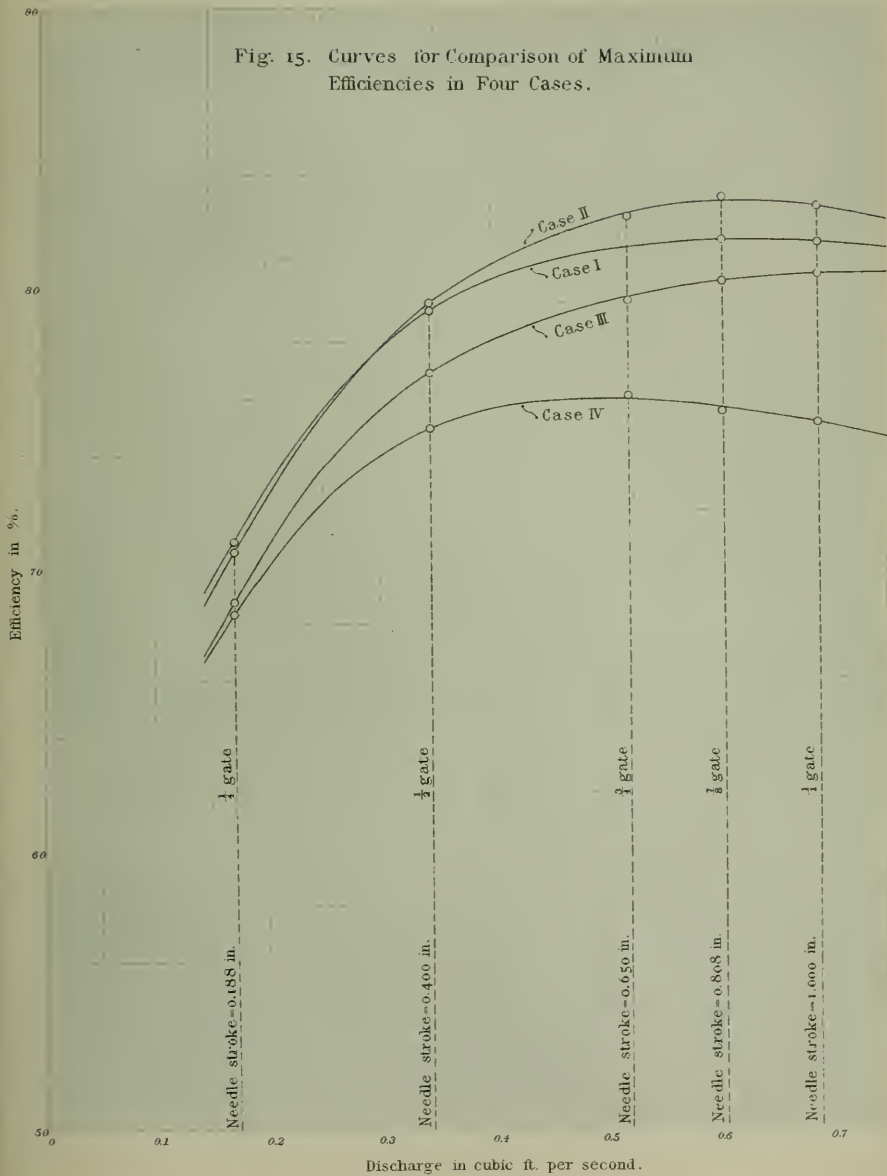
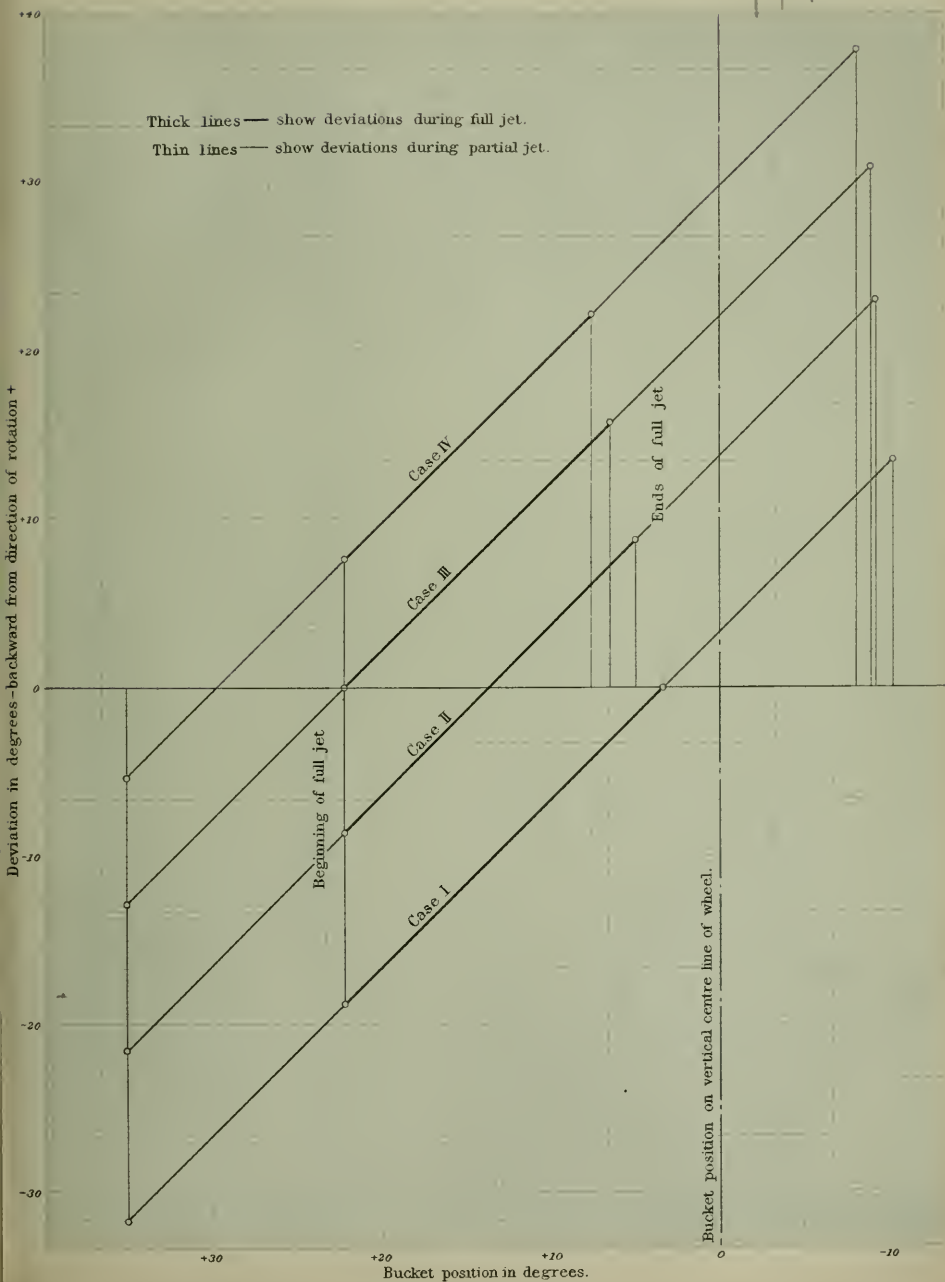


Fig. 16. Deviation of Bucket Plane from
being Normal to Jet Axis.



Elastic Distortions of Framed Structures.

By

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INTRODUCTION.

The problem of elastic distortions of structures is one of the most important studies for structural engineers, not only because it helps to determine deflections, but also because the general principles thereof form the basis for the analysis of statically indeterminate structures.

The distortions of a framed structure may be determined in several ways either analytically or graphically:

1. By the Mohr's work equation—analytically.

2. By the Williot-Mohr diagram—graphically.
3. By the method of elastic loads.
4. By the proposed new method.

Each of these methods has its peculiar merits.

When the deflection of a single panel point or of a small number of panel points is desired the first method is best, being superior to the others in speed as well as precision. But for determination of the displacements of all the panel points this method requires an excessive amount of work.

The advantage of the second method is the completeness of its information. The diagram gives, in a single construction, both the vertical and horizontal displacements of every panel point of the frame. However, in the diagram, there is the cumulative error caused by substituting the tangent of an arc for the arc itself, besides the graphical error which is common with all graphical solutions; so that the solution possesses the disadvantage of lack of precision when compared with the others.

The third method is preferable to the others when the deflection polygon of the frame is desired. If we construct the two deflection polygons of the frame for horizontal and vertical deflections respectively, the actual displacements of all the panel points will be determined. The solution is more precise than that of the Williot-Mohr diagram and more expeditious than that of any other analytical method; and yet the result of the solution is not general, but is only for a case of loadings.

When the general solution of the distortions of the framed structure is required the last method is best. The new method is a complete and exact solution and its determination of the distortions is more general and more precise than that of any other solution. The other advantages of the method are as

follows: The principle is very simple and easily understood. The distortions of the frame can be obtained without determining the stress of each member. Furthermore the new method is equally applicable to the plane truss and the space truss, and likewise to the indeterminate frame as well as to the determinate frame. If we use the method of substitution for the solution, the procedure of the computation may be checked at each step by its control, and the final result can also be verified by Maxwell's law. The general solution obtained by the method is so useful that the displacement influence line of any panel point and the stress influence line of any member of the frame may be determined from the general solution. Consequently, the elastic displacements and the stresses of the frame due to any system of loading can be derived from the same.

Fundamental Principle.

The fundamental principle of the proposed method is as follows: The distortions of a framed structure are the result of changes in the lengths of several members, caused usually by stresses or variations of temperature in the members. If the displacements of the panel points are very small compared with the lengths of the members, the deformations of the members can be expressed by the three rectangular components of the panel point displacements; moreover, under the proportional limit of the material, the deformations of the members are proportional to the internal forces and temperature changes in the members. Therefore the internal forces may be represented by the deformations of the members and variations of temperature, or by the rectangular components of the panel point displacements and temperature changes in the members.

On the other hand, since the framed structure is in stable equilibrium, the condition of equilibrium may evidently be applied to the forces acting upon any portion of the structure. Hence for the internal and external forces acting at a panel point we have three equilibrium equations: i.e., each algebraic sum of the three rectangular components of all the forces acting at the panel point is respectively equal to zero. In this way, as many equations as there are unknown displacement components in the frame may be obtained by extending the equilibrium conditions over all the panel points.

Then solving the equilibrium equations simultaneously, we can determine the rectangular components of all the panel point displacements as linear functions of the given external forces and temperature changes.

In case the effect of temperature is not to be taken into account the required displacement components will be expressed as linear functions of the given external forces.

Elastic Deformations and Internal Forces.

In the analysis of the subject we proceed under the following assumptions which are used in the analysis of framed structures:

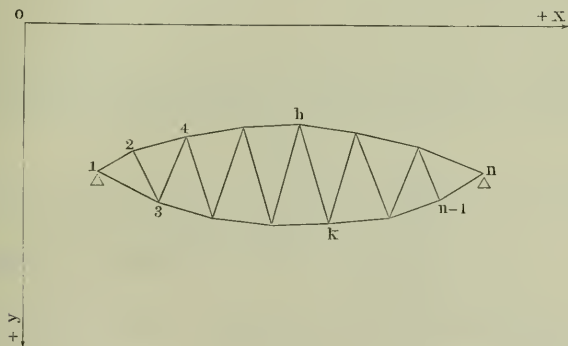
1. All members are straight.
2. The gravity axes of the members meet at panel points.
3. The panel points are frictionless joints.
4. The external forces are applied only at panel points.

Now, let us consider a space framed structure with n panel points (Fig. 1). Let $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots, (x_n, y_n, z_n)$ denote the coordinates of the panel points 1, 2, $\dots$, n , and let $l_{12}, l_{13},$

$\dots, l_{(n-1)n}$ be the lengths of the members, $A_{12}, A_{13}, \dots, A_{(n-1)n}$ the cross sectional areas of the members, and $(\alpha_{12}, \beta_{12}, \gamma_{12}), (\alpha_{13}, \beta_{13}, \gamma_{13}), \dots, (\alpha_{(n-1)n}, \beta_{(n-1)n}, \gamma_{(n-1)n})$ the angles made by the members 1-2, 1-3, $\dots, (n-1)-n$ with the rectangular axes.

If the framed structure is subject to given external forces, the frame will undergo certain distortions which will steadily increase in proportion to the internal forces produced in the members by the external forces gradually applied, and the final deformations will occur in the instant when the external forces are exactly balanced by the internal forces, producing the static equilibrium.

Fig. 1.



Let l_{hk} = length of any member, say $h-k$, when its stress is equal to zero, where k is one of the joints which are directly connected with the panel point h .

S_{hk} =the internal force in the member produced by its deformation.

δl_{hk} =the longitudinal deformation of the member due to its stress S_{hk} , +for extension and -for the contrary.

A_{hk} =the cross sectional area of the member.

E_{hk} =the modulus of elasticity of the material.

While the distortions of the frame are taking place, the length l_{hk} of a member $h-k$ is deformed to $l_{hk} + \delta l_{hk}$; at the same time, the coordinates $(x_h, y_h, z_h), (x_k, y_k, z_k)$ of the panel points h, k are changed to $(x_h + \delta x_h, y_h + \delta y_h, z_h + \delta z_h), (x_k + \delta x_k, y_k + \delta y_k, z_k + \delta z_k)$ respectively (Fig. 2).

Then according to Hooke's law and under the proportional limit of the material

$$\delta l_{hk}/l_{hk} = S_{hk}/E_{hk} \cdot A_{hk} \quad \text{or} \quad \delta l_{hk} = S'_{hk} \cdot l_{hk}/E_{hk} \cdot A_{hk}.$$

In case the effect of temperature is to be taken into account the above value of δl_{hk} must be combined with the quantity

$$\epsilon_{hk} \cdot t_{hk} \cdot l_{hk},$$

in which t_{hk} =a uniform change of temperature of the member expressed in degree, +for rise,

ϵ_{hk} =the coefficient of linear expansion per degree of temperature.

Thus, the most general expression for δl_{hk} is given by the equation

$$\delta l_{hk} = S_{hk} \cdot l_{hk}/E_{hk} \cdot A_{hk} + \epsilon_{hk} \cdot t_{hk} \cdot l_{hk},$$

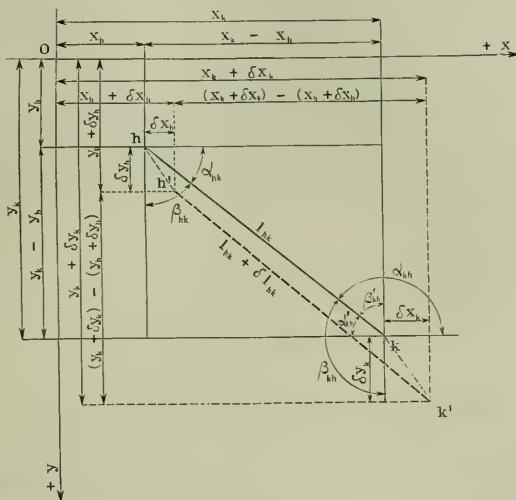
from which we obtain the internal force S_{hk} in the form

$$S_{hk} = (E_{hk} \cdot A_{hk}/l_{hk})(\delta l_{hk} - \epsilon_{hk} \cdot t_{hk} \cdot l_{hk}),$$

where

$$l_{hk} = \{(x_k - x_h)^2 + (y_k - y_h)^2 + (z_k - z_h)^2\}^{\frac{1}{2}}.$$

Fig. 2.



For the new length of the member we have

$$\begin{aligned} (l_{hk} + \delta l_{hk})^2 = & \{(x_k + \delta x_k) - (x_h + \delta x_h)\}^2 \\ & + \{(y_k + \delta y_k) - (y_h + \delta y_h)\}^2 \\ & + \{(z_k + \delta z_k) - (z_h + \delta z_h)\}^2 \end{aligned}$$

or

$$\begin{aligned} 2.l_{hk} \cdot \delta l_{hk} + \delta l_{hk}^2 = & 2(x_k - x_h)(\delta x_k - \delta x_h) + (\delta x_k - \delta x_h)^2 \\ & + 2(y_k - y_h)(\delta y_k - \delta y_h) + (\delta y_k - \delta y_h)^2 \\ & + 2(z_k - z_h)(\delta z_k - \delta z_h) + (\delta z_k - \delta z_h)^2, \end{aligned}$$

where δl_{hk} , δx_k , δy_k , δz_k , δx_h , δy_h , δz_h are very small compared with l_{hk} , x_k , y_k , z_k , x_h , y_h , z_h , so that we may neglect all the product terms of the small quantities.

Hence, since

$$(x_k - x_h)/l_{hk} = \cos \alpha_{hk},$$

$$(y_k - y_h)/l_{hk} = \cos \beta_{hk},$$

$$(z_k - z_h)/l_{hk} = \cos \gamma_{hk},$$

the expression for δl_{hk} becomes simply

$$\delta l_{hk} = (\partial x_k - \partial x_h) \cos \alpha_{hk} + (\partial y_k - \partial y_h) \cos \beta_{hk} + (\partial z_k - \partial z_h) \cos \gamma_{hk}.$$

Substituting this result in the expression of the stress S_{hk} , we have then finally

$$S_{hk} = (E_{hk} A_{hk} / l_{hk}) \left\{ \begin{array}{l} (\partial x_k - \partial x_h) \cos \alpha_{hk} \\ + (\partial y_k - \partial y_h) \cos \beta_{hk} \\ + (\partial z_k - \partial z_h) \cos \gamma_{hk} \end{array} \right\} - E_{hk} A_{hk} \epsilon_{hk}^i l_{hk}^i.$$

Fundamental Equations of Equilibrium.

Since all parts of the framed structure are in equilibrium, the conditions of equilibrium may evidently be applied to the forces acting upon any portion of the structure as well as the structure as a whole. Consequently we can write the equations of equilibrium for the forces acting at each panel point of the frame.

According to the fundamental laws of statics the necessary and sufficient condition of equilibrium for non-coplanar concurrent forces may be expressed thus:

The algebraic sum of the components of all the forces along any axis equals zero,

or, resolving the forces acting at any panel point h into three rectangular components X , Y , Z , we can write the conditions of equilibrium about the panel point in the form

$$\Sigma_h X = 0, \quad \Sigma_h Y = 0, \quad \Sigma_h Z = 0.$$

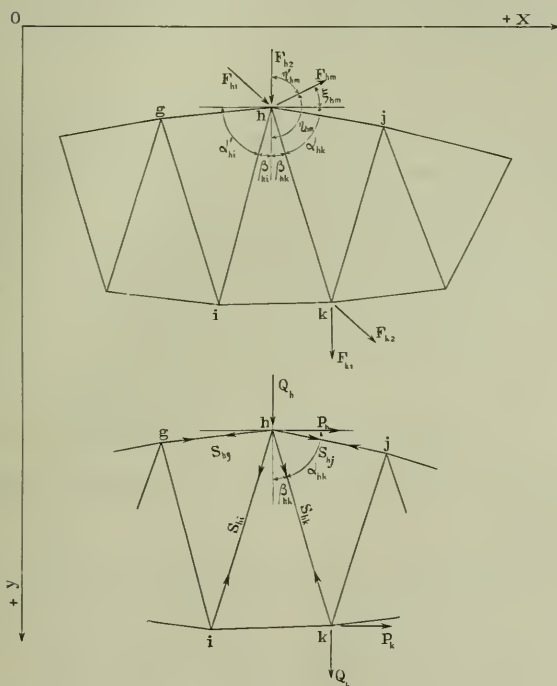
The forces include the internal forces or stresses as well as the external forces. Hence, if F 's denote the external forces and ξ 's, γ 's, ζ 's the direction angles of the action lines (Fig. 3), the condition equations are given in the form

$$\sum_h F_{hm} \cdot \cos \xi_{hm} + \sum_h S_{hk} \cdot \cos \alpha_{hk} = 0,$$

$$\sum_h F_{hm} \cdot \cos \gamma_{hm} + \sum_h S_{hk} \cdot \cos \beta_{hk} = 0,$$

$$\sum_h F_{hm} \cdot \cos \zeta_{hm} + \sum_h S_{hk} \cdot \cos \gamma_{hk} = 0.$$

Fig. 3.



where F'_{hm} is one of the external forces which are exerted at the panel point h .

To simplify the solution we may assume the stress of each member to be tension. Because for the compression member the deformation δl is negative and at the same time the direction of the internal force is just opposite to that of the assumed tension, so either sense of the stress in the member gives the same effect on the condition equations of elastic equilibrium.

Substituting the expression of S_{hk} in the equations of equilibrium, we obtain

$$\Sigma_h \left[(A_{hk} \cdot E_{hk} / l_{hk}) \left\{ \begin{array}{l} (\partial x_k - \delta x_h) \cos \alpha_{hk} \\ + (\partial y_k - \delta y_h) \cos \beta_{hk} \\ + (\partial z_k - \delta z_h) \cos \gamma_{hk} \end{array} \right\} \right] \cos \alpha_{hk} + \Sigma_h F'_{hm} \cos \xi_{hm} - \Sigma_h A_{hk} \cdot E_{hk} \cdot \varepsilon_{hk} \cdot t_{hk} \cos \alpha_{hk} = 0,$$

$$\Sigma_h \left[(A_{hk} \cdot E_{hk} / l_{hk}) \left\{ \begin{array}{l} (\partial x_k - \delta x_h) \cos \alpha_{hk} \\ + (\partial y_k - \delta y_h) \cos \beta_{hk} \\ + (\partial z_k - \delta z_h) \cos \gamma_{hk} \end{array} \right\} \right] \cos \beta_{hk} + \Sigma_h F'_{hm} \cos \eta_{hm} - \Sigma_h A_{hk} \cdot E_{hk} \cdot \varepsilon_{hk} \cdot t_{hk} \cos \beta_{hk} = 0,$$

$$\Sigma_h \left[(A_{hk} \cdot E_{hk} / l_{hk}) \left\{ \begin{array}{l} (\partial x_k - \delta x_h) \cos \alpha_{hk} \\ + (\partial y_k - \delta y_h) \cos \beta_{hk} \\ + (\partial z_k - \delta z_h) \cos \gamma_{hk} \end{array} \right\} \right] \cos \gamma_{hk} + \Sigma_h F'_{hm} \cos \zeta_{hm} - \Sigma_h A_{hk} \cdot E_{hk} \cdot \varepsilon_{hk} \cdot t_{hk} \cos \gamma_{hk} = 0.$$

For the sake of brevity, let us introduce the following abbreviations :

$$\begin{aligned} L_{hk} &= (A_{hk} \cdot E_{hk} / l_{hk}) \cos \alpha_{hk}, & U_h &= \Sigma_h A_{hk} \cdot E_{hk} \cdot \varepsilon_{hk} \cdot t_{hk} \cos \alpha_{hk}, \\ M_{hk} &= (A_{hk} \cdot E_{hk} / l_{hk}) \cos \beta_{hk}, & V_h &= \Sigma_h A_{hk} \cdot E_{hk} \cdot \varepsilon_{hk} \cdot t_{hk} \cos \beta_{hk}, \\ N_{hk} &= (A_{hk} \cdot E_{hk} / l_{hk}) \cos \gamma_{hk}, & W_h &= \Sigma_h A_{hk} \cdot E_{hk} \cdot \varepsilon_{hk} \cdot t_{hk} \cos \gamma_{hk}, \\ P_h &= \Sigma_h F'_{hm} \cos \xi_{hm}, & p_h &= \partial x_h, \quad p_k = \partial x_k, \end{aligned}$$

$$Q_h = \Sigma_h F_{hm} \cos \eta_{hm}, \quad q_h = \delta y_h, \quad q_k = \delta y_k,$$

$$R_h = \Sigma_h F_{hm} \cos \zeta_{hm}, \quad r_h = \delta z_h, \quad r_k = \delta z_k.$$

Then we can write the conditions of equilibrium in the form

$$\Sigma_h [L_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} + (r_k - r_h) \cos \gamma_{hk} \}] = U_h - P_h,$$

$$\Sigma_h [M_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} + (r_k - r_h) \cos \gamma_{hk} \}] = V_h - Q_h,$$

$$\Sigma_h [N_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} + (r_k - r_h) \cos \gamma_{hk} \}] = W_h - R_h,$$

in which L_{hk} , M_{hk} , N_{hk} , U_h , V_h , W_h , P_h , Q_h , R_h may be obtained from the given data, while p_h , q_h , r_h , p_k , q_k , r_k are unknown displacements.

Thus we have three conditional equations of elastic equilibrium about each panel point of the frame; and extending the conditions of equilibrium over all the panel points we obtain as many equations as there are unknown displacements: that is, there are $3n$ linear equations for $3n$ unknowns ($p's$, $q's$, $r's$).

Moreover, in the case of a plane truss, the conditional equations of elastic equilibrium for any panel point h of the frame can be expressed in the form:

$$\Sigma_h [L_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} \}] = U_h - P_h,$$

$$\Sigma_h [M_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} \}] = V_h - Q_h.$$

Similar equations may be written out for each of the other panel points of the plane frame, and we obtain $2n$ linear equations for $2n$ unknown displacement components of the panel points.

Hence solving simultaneously the linear equations we can determine the values of all the unknown displacements of the panel points.

Solution of the Equations of Equilibrium.

When the equations of elastic equilibrium have been written down, then they form a series of simultaneous linear equations, from which the unknown elastic displacements can be determined.

The solution of the equations is only arithmetic operations and is somewhat lengthy but not difficult.

For simplicity, neglecting the temperature effect, we can write the conditional equations for any panel point h of a frame in the form:

$$\Sigma_h [L_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} + (r_k - r_h) \cos \gamma_{hk} \}] = -P_h,$$

$$\Sigma_h [M_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} + (r_k - r_h) \cos \gamma_{hk} \}] = -Q_h,$$

$$\Sigma_h [N_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} + (r_k - r_h) \cos \gamma_{hk} \}] = -R_h,$$

or, in the general form

$$a_{11}p_1 + a_{12}p_2 + \dots + a_{1n}p_n + d_{11}q_1 + \dots + d_{1n}q_n + e_{11}r_1 + \dots + e_{1n}r_n = -P_1,$$

$$a_{21}p_1 + a_{22}p_2 + \dots + a_{2n}p_n + d_{21}q_1 + \dots + d_{2n}q_n + e_{21}r_1 + \dots + e_{2n}r_n = -P_2,$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$a_{n1}p_1 + a_{n2}p_2 + \dots + a_{nn}p_n + d_{n1}q_1 + \dots + d_{nn}q_n + e_{n1}r_1 + \dots + e_{nn}r_n = -P_n,$$

$$d_{11}p_1 + d_{12}p_2 + \dots + d_{1n}p_n + b_{11}q_1 + \dots + b_{1n}q_n + f_{11}r_1 + \dots + f_{1n}r_n = -Q_1,$$

$$d_{21}p_1 + d_{22}p_2 + \dots + d_{2n}p_n + b_{21}q_1 + \dots + b_{2n}q_n + f_{21}r_1 + \dots + f_{2n}r_n = -Q_2,$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$d_{n1}p_1 + d_{n2}p_2 + \dots + d_{nn}p_n + b_{n1}q_1 + \dots + b_{nn}q_n + f_{n1}r_1 + \dots + f_{nn}r_n = -Q_n,$$

$$e_{11}p_1 + e_{12}p_2 + \dots + e_{1n}p_n + f_{11}q_1 + \dots + f_{1n}q_n + c_{11}r_1 + \dots + c_{1n}r_n = -R_1,$$

$$e_{21}p_1 + e_{22}p_2 + \dots + e_{2n}p_n + f_{21}q_1 + \dots + f_{2n}q_n + c_{21}r_1 + \dots + c_{2n}r_n = -R_2,$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$e_{n1}p_1 + e_{n2}p_2 + \dots + e_{nn}p_n + f_{n1}q_1 + \dots + f_{nn}q_n + c_{n1}r_1 + \dots + c_{nn}r_n = -R_n.$$

in which the coefficients $a's$, $b's$, $c's$, $d's$, $e's$, $f's$ are independent of the loading, but depend upon the geometry of the structure and the modulus of elasticity of the material.

The simultaneous equations may be solved by the ordinary algebraic methods for the elimination of linear equations, by the method of determinants, or by the method of substitution.

First solution: Method of determinants.

Solving the $3n$ linear simultaneous equations by the method of determinants we obtain the $3n$ unknown displacements $p's$, $q's$, $r's$ as follows:

$$\begin{aligned} p_1 &= D_1/D, & p_2 &= D_2/D, & \dots, & p_n &= D_n/D, \\ q_1 &= D_{n+1}/D, & q_2 &= D_{n+2}/D, & \dots, & q_n &= D_{2n}/D, \\ r_1 &= D_{2n+1}/D, & r_2 &= D_{2n+2}/D, & \dots, & r_n &= D_{3n}/D, \end{aligned}$$

where

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & d_{11} & \dots & d_{1n} & e_{11} & \dots & e_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} & d_{21} & \dots & d_{2n} & e_{21} & \dots & e_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & d_{n1} & \dots & d_{nn} & e_{n1} & \dots & e_{nn} \\ d_{11} & d_{12} & \dots & d_{1n} & b_{11} & \dots & b_{1n} & f_{11} & \dots & f_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ d_{n1} & d_{n2} & \dots & d_{nn} & b_{n1} & \dots & b_{nn} & f_{n1} & \dots & f_{nn} \\ e_{11} & e_{12} & \dots & e_{1n} & f_{11} & \dots & f_{1n} & c_{11} & \dots & c_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ e_{n1} & e_{n2} & \dots & e_{nn} & f_{n1} & \dots & f_{nn} & c_{n1} & \dots & c_{nn} \end{vmatrix}$$

and since

$$a_{st} = a_{ts}, \quad b_{st} = b_{ts}, \quad c_{st} = c_{ts}, \quad d_{st} = d_{ts}, \quad e_{st} = e_{ts}, \quad f_{st} = f_{ts},$$

the determinant D is of the symmetrical type; and D_s is the $3n$ -rowed determinant obtained from D by replacing the elements of the s^{th} column by the elements

$$-P_1, -P_2, \dots, -P_n, -Q_1, \dots, -Q_n, -R_1, \dots, -R_n.$$

Second solution: Method of substitution.

When the linear equations are numerous, the method of substitution is more suitable. For convenience, taking three unknowns, p, q, r , (the process being the same whatever the number) we have the simultaneous linear equations in the form

$$a.p + d.q + e.r = -P,$$

$$d.p + b.q + f.r = -Q,$$

$$e.p + f.q + c.r = -R.$$

Solving the equations by the method of substitution, we obtain

$$r = -(R.2)/(c.2),$$

$$q = -\{(f.1)/(b.1)\}r - (Q.1)/(b.1),$$

$$p = -(d/a).q - (e/a)r - (P/a),$$

where

$$(b.1) = b - (d/a).d,$$

$$(f.1) = f - (d/a).e,$$

$$(c.1) = c - (e/a).e,$$

$$(c.2) = (c.1) - \{(f.1)/(b.1)\}(f.1),$$

$$(Q.1) = Q - (d/a).P,$$

$$(R.1) = R - (e/a).P.$$

$$(R.2) = (R.1) - \{(f.1)/(b.1)\}(Q.1).$$

In applying the method of substitution to any numerical

example, it is convenient to arrange the computation as in the following form. It is well fitted for use with the calculating machine or slide rule.

Given the simultaneous equations :

No.	p	q	r		Check	Remarks
(1)	a	d	e	P	(as)	
(2)	d	b	f	Q	(bs)	
(3)	e	f	c	R	(cs)	
Solution.						
(4)	1	d/a	e/a	P/a	$(as)/a$	$(1)/a$
(5)	d	$d.d/a$	$d.e/a$	$d.P/a$	$d.(as)/a$	$(4) \times d$
(2)	d	b	f	Q	(bs)	(2)
(6)		$(b.1)$	$(f.1)$	$(Q.1)$	$(bs.1)$	$(2) - (5)$
(7)	e	$e.d/a$	$e.e/a$	$e.P/a$	$e.(as)/a$	$(4) \times e$
(3)	e	f	c	R	(cs)	(3)
(8)		$(f.1)$	$(c.1)$	$(R.1)$	$(cs.1)$	$(3) - (7)$
(9)		1	$\frac{(f.1)}{(b.1)}$	$\frac{(Q.1)}{(b.1)}$	$\frac{(bs.1)}{(b.1)}$	$\frac{(6)}{(b.1)}$
(10)		$(f.1)$	$\frac{(f.1)(f.1)}{(b.1)}$	$\frac{(f.1)(Q.1)}{(b.1)}$	$\frac{(f.1)(bs.1)}{(b.1)}$	$(9) \times (f.1)$
(8)		$(f.1)$	$(c.1)$	$(R.1)$	$(cs.1)$	(8)
(11)			$(c.2)$	$(R.2)$	$(cs.2)$	$(8) - (10)$
(12)			1	$\frac{(R.2)}{(c.2)}$	$\frac{(cs.2)}{(c.2)}$	$\frac{(11)}{(c.2)}$

$$A_{st} = A_{ts}, \quad B_{st} = B_{ts}, \quad C_{st} = C_{ts},$$

$$D_{st} = D_{ts}, \quad E_{st} = E_{ts}, \quad F_{st} = F_{ts},$$

which may be applied as the check of the solution.

Applications of the Proposed Method.

The following examples are given in order to show the use of the proposed method for finding the general solution of the elastic distortions of framed structures, and also to illustrate the application of the method for finding the general solution of the stresses of the members.

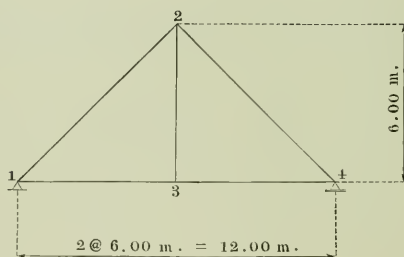
Example 1. To determine the general solution of the elastic distortions of a simple truss shown in Fig. 4, assuming the modulus of elasticity $E=2,000,000$ kg. per sq. cm.

Given data are as follows:

Member.	Length. l (cm.)	Area. A (cm. ²)	E.A/l. μ (kg./cm.)	$\cos \alpha$.	$\cos \beta$.
1-2	$600\sqrt{2}$	$30\sqrt{2}$	10^5	$1/\sqrt{2}$	$-1/\sqrt{2}$
1-3	600	30	10^5	1	0
2-3	600	30	10^5	0	1
2-4	$600\sqrt{2}$	$30\sqrt{2}$	10^5	$1/\sqrt{2}$	$1/\sqrt{2}$
3-4	600	30	10^5	1	0

Member.	$L =$ $\mu \cos \alpha$ (kg./cm.)	$M =$ $\mu \cos \beta$ (kg./cm.)	$L \cos \alpha$ (kg./cm.)	$L \cos \beta$ (kg./cm.)	$M \cos \alpha$ (kg./cm.)	$M \cos \beta$ (kg./cm.)
1-2	$10^5/\sqrt{2}$	$-10^5/\sqrt{2}$	$10^5/2$	$-10^5/2$	$-10^5/2$	$10^5/2$
1-3	10^5	0	10^5	0	0	0
2-3	0	10^5	0	0	0	10^5
2-4	$10^5/\sqrt{2}$	$10^5/\sqrt{2}$	$10^5/2$	$10^5/2$	$10^5/2$	$10^5/2$
3-4	10^5	0	10^5	0	0	0

Fig. 4.



The general conditions of elastic equilibrium for any panel point (h) of a plane frame are

$$\Sigma X_h = 0, \text{ or } \Sigma_h [L_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} \}] = -P_h,$$

$$\Sigma Y_h = 0, \text{ or } \Sigma_h [M_{hk} \{(\rho_k - \rho_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk}\}] = -Q_h,$$

where

$$L_{hk} = (F_{hk} \cdot A_{hk} / l_{hk}) \cos \alpha_{hk}, \quad M_{hk} = (F_{hk} \cdot A_{hk} / l_{hk}) \cos \beta_{hk}.$$

The reaction conditions of the given framed structure are

$$p_1 = 0, \quad q_1 = 0, \quad \text{and} \quad q_4 = 0.$$

Then apply the condition of $\Sigma X = 0$ to each panel point:

At the panel point (2)

$$\begin{aligned} \Sigma X_2 = 0, \text{ or } \quad & L_{21} \{(\rho_1 - \rho_2) \cos \alpha_{21} + (q_1 - q_2) \cos \beta_{21}\} \\ & + L_{23} \{(\rho_3 - \rho_2) \cos \alpha_{23} + (q_3 - q_2) \cos \beta_{23}\} \\ & + L_{24} \{(\rho_4 - \rho_2) \cos \alpha_{24} + (q_4 - q_2) \cos \beta_{24}\} = -P_2, \end{aligned}$$

or inserting the values given in the table we have

$$-\rho_2 + \frac{1}{2}\rho_4 = -P_2(10^{-5}).$$

Similarly we obtain at the panel points (3) and (4)

$$-2\rho_3 + \rho_4 = -P_3(10^{-5}),$$

$$\frac{1}{2}\rho_2 + \rho_3 - (3/2)\rho_4 + \frac{1}{2}q_2 = -P_4(10^{-5}).$$

Secondly apply the condition of $\Sigma Y = 0$ to each panel point:

At the panel point (2)

$$\begin{aligned} \Sigma Y_2 = 0, \text{ or } \quad & M_{21} \{(\rho_1 - \rho_2) \cos \alpha_{21} + (q_1 - q_2) \cos \beta_{21}\} \\ & + M_{23} \{(\rho_3 - \rho_2) \cos \alpha_{23} + (q_3 - q_2) \cos \beta_{23}\} \\ & + M_{24} \{(\rho_4 - \rho_2) \cos \alpha_{24} + (q_4 - q_2) \cos \beta_{24}\} = -Q_2, \end{aligned}$$

or inserting the values given in the table we have

$$\frac{1}{2}\rho_4 - 2q_2 + q_3 = -Q_2(10^{-5}).$$

Similarly we obtain at the panel point (3)

$$q_2 - q_3 = -Q_3(10^{-5}).$$

Thus for the conditional equations we obtain

$$\begin{aligned}
 -(1)p_2 + (0)p_3 + (1/2)p_4 + (0)q_2 + (0)q_3 &= -P_2(10^{-5}), \\
 +(0)p_2 - (2)p_3 + (1)p_4 + (0)q_2 + (0)q_3 &= -P_3(10^{-5}), \\
 +(1/2)p_2 + (1)p_3 - (3/2)p_4 + (1/2)q_2 + (0)q_3 &= -P_4(10^{-5}), \\
 +(0)p_2 + (0)p_3 + (1/2)p_4 - (2)q_2 + (1)q_3 &= -Q_2(10^{-5}), \\
 +(0)p_2 + (0)p_3 + (0)p_4 + (1)q_2 - (1)q_3 &= -Q_3(10^{-5}).
 \end{aligned}$$

Solving the linear simultaneous equations we obtain the general solution of the displacement components of the panel points of the frame in the form

$$\begin{aligned}
 p_2(10^5) &= (3/2)P_2 + (1/2)P_3 + (1)P_4 + (1/2)Q_2 + (1/2)Q_3, \\
 p_3(10^5) &= (1/2)P_2 + (1)P_3 + (1)P_4 + (1/2)Q_2 + (1/2)Q_3, \\
 p_4(10^5) &= (1)P_2 + (1)P_3 + (2)P_4 + (1)Q_2 + (1)Q_3, \\
 q_2(10^5) &= (1/2)P_2 + (1/2)P_3 + (1)P_4 + (3/2)Q_2 + (3/2)Q_3, \\
 q_3(10^5) &= (1/2)P_2 + (1/2)P_3 + (1)P_4 + (3/2)Q_2 + (5/2)Q_3.
 \end{aligned}$$

From this general solution we can easily determine the displacement components of the panel points of the frame due to any system of loading.

First, if

$$P_2=0, \quad P_3=0, \quad P_4=0,$$

$$Q_2=353 \text{ kg.},$$

$$Q_3=212 \text{ kg.},$$

that is, the case of the dead load, we obtain the displacement components of the panel points as follows:

$$p_1=0, \quad q_1=0,$$

$$p_2=0.003 \text{ cm.}, \quad q_2=0.008 \text{ cm.},$$

$$p_3=0.003 \text{ cm.}, \quad q_3=0.011 \text{ cm.},$$

$$p_4=0.006 \text{ cm.}, \quad q_4=0.$$

Secondly, if

$$P_2=0, \quad P_3=0, \quad P_4=0, \quad Q_2=0,$$

$$Q_3=10,000 \text{ kg.},$$

that is, a case of live loads, we get the displacements as follows:

$$p_1=0, \quad q_1=0,$$

$$p_2=0.050 \text{ cm.}, \quad q_2=0.150 \text{ cm.},$$

$$p_3=0.050 \text{ cm.}, \quad q_3=0.250 \text{ cm.},$$

$$p_4=0.100 \text{ cm.}, \quad q_4=0.$$

Thirdly, if

$$P_2=1,000 \text{ kg.}, \quad P_3=0, \quad P_4=0,$$

$$Q_2=0, \quad Q_3=0,$$

that is, the case of a horizontal load, we have the displacements as follows:

$$p_1=0, \quad q_1=0,$$

$$p_2=0.015 \text{ cm.}, \quad q_2=0.005 \text{ cm.},$$

$$p_3=0.005 \text{ cm.}, \quad q_3=0.005 \text{ cm.},$$

$$p_4=0.010 \text{ cm.}, \quad q_4=0.$$

Fourthly, if

$$P_2=0, \quad P_3=0, \quad P_4=-1,000 \text{ kg.},$$

$$Q_2=0, \quad Q_3=0,$$

for the displacement components of the panel points we get

$$p_1=0, \quad q_1=0,$$

$$p_2=-0.010 \text{ cm.}, \quad q_2=-0.010 \text{ cm.},$$

$$p_3 = -0.010 \text{ cm.}, \quad q_3 = -0.010 \text{ cm.},$$

$$p_4 = -0.020 \text{ cm.}, \quad q_4 = 0.$$

Fifthly, if

$$P_2 = 1,000 \text{ kg.}, \quad P_3 = 0, \quad P_4 = 0,$$

$$Q_2 = 353 \text{ kg.},$$

$$Q_3 = 212 + 10,000 = 10,212 \text{ kg.},$$

that is, a case of simultaneous loadings, for the displacement components of the panel points we obtain

$$p_1 = 0, \quad q_1 = 0,$$

$$p_2 = 0.068 \text{ cm.}, \quad q_2 = 0.163 \text{ cm.},$$

$$p_3 = 0.058 \text{ cm.}, \quad q_3 = 0.266 \text{ cm.},$$

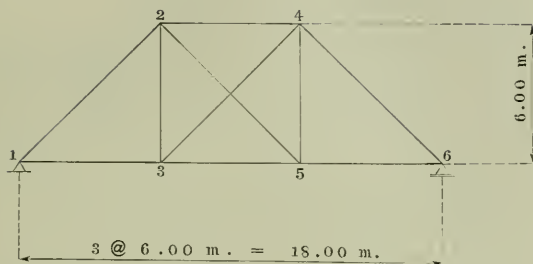
$$p_4 = 0.116 \text{ cm.}, \quad q_4 = 0.$$

Example 2. To determine the complete general solution of the redundant framed structure as shown in Fig. 5, taking the modulus of elasticity of the material $E = 2,000,000$ kg. per sq. cm.

Given data are as follows:

Member.	Length. l (cm.)	Area. A (cm. ²)	E.A/l. μ (kg./cm.)	cos α .	cos β .
1-2	$600\sqrt{2}$	$30\sqrt{2}$	10^5	$1/\sqrt{2}$	$-1/\sqrt{2}$
1-3	600	30	10^5	1	0
2-3	600	30	10^5	0	1
2-4	600	30	10^5	1	0
2-5	$600\sqrt{2}$	$30\sqrt{2}$	10^5	$1/\sqrt{2}$	$1/\sqrt{2}$
3-4	$600\sqrt{2}$	$30\sqrt{2}$	10^5	$1/\sqrt{2}$	$-1/\sqrt{2}$
3-5	600	30	10^5	1	0
4-5	600	30	10^5	0	1
4-6	$600\sqrt{2}$	$30\sqrt{2}$	10^5	$1/\sqrt{2}$	$1/\sqrt{2}$
5-6	600	30	10^5	1	0

Fig. 5.



Member.	$L = \mu \cos \alpha$ (kg./cm.)	$M = \mu \cos \beta$ (kg./cm.)	$I_L \cos \alpha$ (kg./cm.)	$I_L \cos \beta$ (kg./cm.)	$M \cos \alpha$ (kg./cm.)	$M \cos \beta$ (kg./cm.)
1-2	$10^5/\sqrt{2}$	$-10^5/\sqrt{2}$	$10^5/2$	$-10^5/2$	$-10^5/2$	$10^5/2$
1-3	10^5	0	10^5	0	0	0
2-3	0	10^5	0	0	0	10^5
2-4	10^5	0	10^5	0	0	0
2-5	$10^5/\sqrt{2}$	$10^5/\sqrt{2}$	$10^5/2$	$10^5/2$	$10^5/2$	$10^5/2$
3-4	$10^5/\sqrt{2}$	$-10^5/\sqrt{2}$	$10^5/2$	$-10^5/2$	$-10^5/2$	$10^5/2$
3-5	10^5	0	10^5	0	0	0
4-5	0	10^5	0	0	0	10^5
4-6	$10^5/\sqrt{2}$	$10^5/\sqrt{2}$	$10^5/2$	$10^5/2$	$10^5/2$	$10^5/2$
5-6	10^5	0	10^5	0	0	0

First determine the general solutions of the elastic distortions of the frame, and then find the general expressions of the stresses in all the members.

The general equations of elastic equilibrium for any panel point (h) of a plane frame are

$$\Sigma X_h = 0, \text{ or } \Sigma_h [L_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} \}] = -P_h,$$

$$\Sigma Y_h = 0, \text{ or } \Sigma_h [M_{hk} \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} \}] = -Q_h,$$

where

$$L_{hk} = (E_{hk} \cdot A_{hk} / l_{hk}) \cos \alpha_{hk}, \quad M_{hk} = (E_{hk} \cdot A_{hk} / l_{hk}) \cos \beta_{hk}.$$

The reaction conditions of the given truss are

$$p_1 = 0, \quad q_1 = 0, \quad \text{and} \quad q_6 = 0.$$

Then apply the condition of $\Sigma X = 0$ to each panel point:

At the panel point (2)

$$\begin{aligned} \Sigma X_2 = 0, \text{ or } & L_{21} \{ (p_1 - p_2) \cos \alpha_{21} + (q_1 - q_2) \cos \beta_{21} \} \\ & + L_{23} \{ (p_3 - p_2) \cos \alpha_{23} + (q_3 - q_2) \cos \beta_{23} \} \\ & + L_{24} \{ (p_4 - p_2) \cos \alpha_{24} + (q_4 - q_2) \cos \beta_{24} \} \\ & + L_{25} \{ (p_5 - p_2) \cos \alpha_{25} + (q_5 - q_2) \cos \beta_{25} \} = -P_2, \end{aligned}$$

or inserting the values given in the table, we have

$$-(2)p_2 + (1)p_4 + (1/2)p_5 + (1/2)q_5 = -P_2(10^{-5}).$$

Similarly we obtain at the panel points (3), (4), (5) and (6)

$$-(5/2)p_3 + (1/2)p_4 + (1)p_5 + (1/2)q_5 - (1/2)q_4 = -P_3(10^{-5}),$$

$$(1)p_2 + (1/2)p_5 - (2)p_4 + (1/2)p_6 - (1/2)q_5 = -P_4(10^{-5}),$$

$$(1/2)p_2 + (1)p_5 - (5/2)p_5 + (1)p_6 + (1/2)q_2 - (1/2)q_5 = -P_5(10^{-5}),$$

$$(1/2)p_4 + (1)p_5 - (3/2)p_6 + (1/2)q_4 = -P_6(10^{-5}).$$

Secondly apply the condition of $\Sigma Y = 0$ to each panel point:

At the panel point (2)

$$\begin{aligned} \Sigma Y_2=0, \text{ or } & M_{21}\{(\rho_1-\rho_2) \cos \alpha_{21}+(q_1-q_2) \cos \beta_{21}\} \\ & +M_{23}\{(\rho_3-\rho_2) \cos \alpha_{23}+(q_3-q_2) \cos \beta_{23}\} \\ & +M_{24}\{(\rho_4-\rho_2) \cos \alpha_{24}+(q_4-q_2) \cos \beta_{24}\} \\ & +M_{25}\{(\rho_5-\rho_2) \cos \alpha_{25}+(q_5-q_2) \cos \beta_{25}\}=-Q_2, \end{aligned}$$

or inserting the values given in the table, we have

$$(1/2)\rho_5-(2)q_2+(1)q_3+(1/2)q_5=-Q_2(10^{-5}).$$

Similarly we obtain at the panel points (3), (4) and (5)

$$\begin{aligned} (1/2)\rho_3-(1/2)\rho_4+(1)q_2-(3/2)q_3+(1/2)q_4 &=-Q_3(10^{-5}), \\ -(1/2)\rho_3+(1/2)\rho_6+(1/2)q_3-(2)q_4+(1)q_5 &=-Q_4(10^{-5}), \\ (1/2)\rho_2-(1/2)\rho_5+(1/2)q_2+(1)q_4-(3/2)q_5 &=-Q_5(10^{-5}). \end{aligned}$$

Thus for the conditions of the elastic equilibrium of the redundant frame we have the following 9 equations:

$$\begin{aligned} -P_2(10^{-5}) &=-(2)\rho_2+(0)\rho_3+(1)\rho_4+(1/2)\rho_5+(0)\rho_6+(0)q_2+(0)q_3+(0)q_4+(1/2)q_5, \\ -P_3(10^{-5}) &=(0)\rho_2-(5/2)\rho_3+(1/2)\rho_4+(1)\rho_5+(0)\rho_6+(0)q_2+(1/2)q_3-(1/2)q_4+(0)q_5, \\ -P_4(10^{-5}) &=(1)\rho_2+(1/2)\rho_3-(2)\rho_4+(0)\rho_5+(1/2)\rho_6+(0)q_2-(1/2)q_3+(0)q_4+(0)q_5, \\ -P_5(10^{-5}) &=(1/2)\rho_2+(1)\rho_3+(0)\rho_4-(5/2)\rho_5+(1)\rho_6+(1/2)q_2+(0)q_3+(0)q_4-(1/2)q_5, \\ -P_6(10^{-5}) &=(0)\rho_2+(0)\rho_3+(1/2)\rho_4+(1)\rho_5-(3/2)\rho_6+(0)q_2+(0)q_3+(1/2)q_4+(0)q_5, \\ -Q_2(10^{-5}) &=(0)\rho_2+(0)\rho_3+(0)\rho_4+(1/2)\rho_5+(0)\rho_6-(2)q_2+(1)q_3+(0)q_4+(1/2)q_5, \\ -Q_3(10^{-5}) &=(0)\rho_2+(1/2)\rho_3-(1/2)\rho_4+(0)\rho_5+(0)\rho_6+(1)q_2-(3/2)q_3+(1/2)q_4+(0)q_5, \\ -Q_4(10^{-5}) &=(0)\rho_2-(1/2)\rho_3+(0)\rho_4+(0)\rho_5+(1/2)\rho_6+(0)q_2+(1/2)q_3-(2)q_4+(1)q_5, \\ -Q_5(10^{-5}) &=(1/2)\rho_2+(0)\rho_3+(0)\rho_4-(1/2)\rho_5+(0)\rho_6+(1/2)q_2+(0)q_3+(1)q_4-(3/2)q_5, \end{aligned}$$

Solving the linear simultaneous equations of $(p's, q's)$ we obtain finally the general solutions of the displacement components of all the panel points in terms of the external forces $(P's, Q's)$ as given in the following Table A.

TABLE A.

$\mu_2(10^5) = 1.666\dot{6}P_2 + 0.666\dot{6}P_3 + 1.166\dot{6}P_4 + 1.166\dot{6}P_5 + 1.5000P_6 + 1.0000Q_2 + 0.833\dot{3}Q_3 + 1.0000Q_4 + 1.166\dot{6}Q_5,$
$\mu_3(10^5) = 0.666\dot{6}P_2 + 1.0000P_3 + 0.666\dot{6}P_4 + 1.0000P_5 + 1.0000P_6 + 0.666\dot{6}Q_2 + 0.666\dot{6}Q_3 + 0.333\dot{3}Q_4 + 0.333\dot{3}Q_5,$
$\mu_4(10^5) = 1.166\dot{6}P_2 + 0.666\dot{6}P_3 + 1.541\dot{6}P_4 + 1.041\dot{6}P_5 + 1.3750P_6 + 0.5000Q_2 + 0.208\dot{3}Q_3 + 0.5000Q_4 + 0.541\dot{6}Q_5,$
$\mu_5(10^5) = 1.166\dot{6}P_2 + 1.0000P_3 + 1.041\dot{6}P_4 + 1.8750P_5 + 1.8750P_6 + 1.166\dot{6}Q_2 + 1.041\dot{6}Q_3 + 0.833\dot{3}Q_4 + 0.708\dot{3}Q_5,$
$\mu_6(10^5) = 1.5000P_2 + 1.0000P_3 + 1.3750P_4 + 1.8750P_5 + 2.8750P_6 + 1.5000Q_2 + 1.3750Q_3 + 1.5000Q_4 + 1.3750Q_5,$
$q_2(10^5) = 1.0000P_2 + 0.666\dot{6}P_3 + 0.5000P_4 + 1.166\dot{6}P_5 + 1.5000P_6 + 2.333\dot{3}Q_2 + 2.333\dot{3}Q_3 + 1.666\dot{6}Q_4 + 1.833\dot{3}Q_5,$
$q_3(10^5) = 0.833\dot{3}P_2 + 0.666\dot{6}P_3 + 0.208\dot{3}P_4 + 1.041\dot{6}P_5 + 1.3750P_6 + 2.166\dot{6}Q_2 + 2.8750Q_3 + 1.833\dot{3}Q_4 + 1.8750Q_5,$
$q_4(10^5) = 1.0000P_2 + 0.333\dot{3}P_3 + 0.5000P_4 + 0.833\dot{3}P_5 + 1.5000P_6 + 1.666\dot{6}Q_2 + 1.833\dot{3}Q_3 + 2.333\dot{3}Q_4 + 2.166\dot{6}Q_5,$
$q_5(10^5) = 1.166\dot{6}P_2 + 0.333\dot{3}P_3 + 0.541\dot{6}P_4 + 0.708\dot{3}P_5 + 1.3750P_6 + 1.833\dot{3}Q_2 + 1.8750Q_3 + 2.166\dot{6}Q_4 + 2.8750Q_5.$

Next, we shall determine the general solution of the stress of each member of the given frame. Since the fundamental equation of an elastic condition for a member $h-k$ of a plane frame is

$$\begin{aligned}\delta l_{hk} &= S_{hk} \cdot l_{hk} / A_{hk} \cdot E_{hk} + \epsilon_{hk} \cdot t_{hk} \cdot l_{hk} \\ &= (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk},\end{aligned}$$

the general expression of the stress of the member may be expressed in the form

$$S_{hk} = (A_{hk} \cdot E_{hk} / l_{hk}) \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} \} - A_{hk} \cdot E_{hk} \cdot \epsilon_{hk} \cdot t_{hk}.$$

When temperature effect is neglected the expression of the stress becomes

$$\begin{aligned}S_{hk} &= (A_{hk} \cdot E_{hk} / l_{hk}) \{ (p_k - p_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} \} \\ &= L_{hk} (p_k - p_h) + M_{hk} (q_k - q_h).\end{aligned}$$

By substituting the general expression of each displacement component in the above equation we can determine the general solutions of the stresses of all the members as linear functions of the external forces ($P's$, $Q's$).

In the present problem, for the stress of the member 1-2 we have

$$S_{12} = L_{12} (p_2 - p_1) + M_{12} (q_2 - q_1),$$

where

$$L_{12} = 10^5 / \sqrt{2}, \quad M_{12} = -10^5 / \sqrt{2},$$

$$p_1 = 0, \quad q_1 = 0,$$

$$\begin{aligned}p_2 &= (1.66\dot{6} P_2 + 0.66\dot{6} P_3 + 1.16\dot{6} P_4 + 1.16\dot{6} P_5 + 1.500 P_6 \\ &\quad + 1.000 Q_2 + 0.83\dot{3} Q_3 + 1.000 Q_4 + 1.16\dot{6} Q_5) (10^{-5}),\end{aligned}$$

$$q_2 = (1.000 P_2 + 0.66\dot{6} P_3 + 0.500 P_4 + 1.16\dot{6} P_5 + 1.500 P_6$$

$$+ 2.33\dot{3}Q_2 + 2.16\dot{6}Q_3 + 1.66\dot{6}Q_4 + 1.83\dot{3}Q_5)(10^{-5}).$$

Substituting these values in the equation of the stress S_{12} we get

$$\begin{aligned} S'_{12} &= (p_2 - q_2)(10^5/\sqrt{2}) \\ &= (0.66\dot{6}P_2 + 0.66\dot{6}P_4 - 1.33\dot{3}Q_2 - 1.33\dot{3}Q_3 \\ &\quad - 0.66\dot{6}Q_4 - 0.66\dot{6}Q_5)(1/\sqrt{2}) \\ &= 0.471,405P_2 + 0.471,405P_4 - 0.942,809Q_2 - 0.942,809Q_3 \\ &\quad - 0.471,405Q_4 - 0.471,405Q_5. \end{aligned}$$

Similarly we obtain the general solutions of the stresses of the other members as follows :

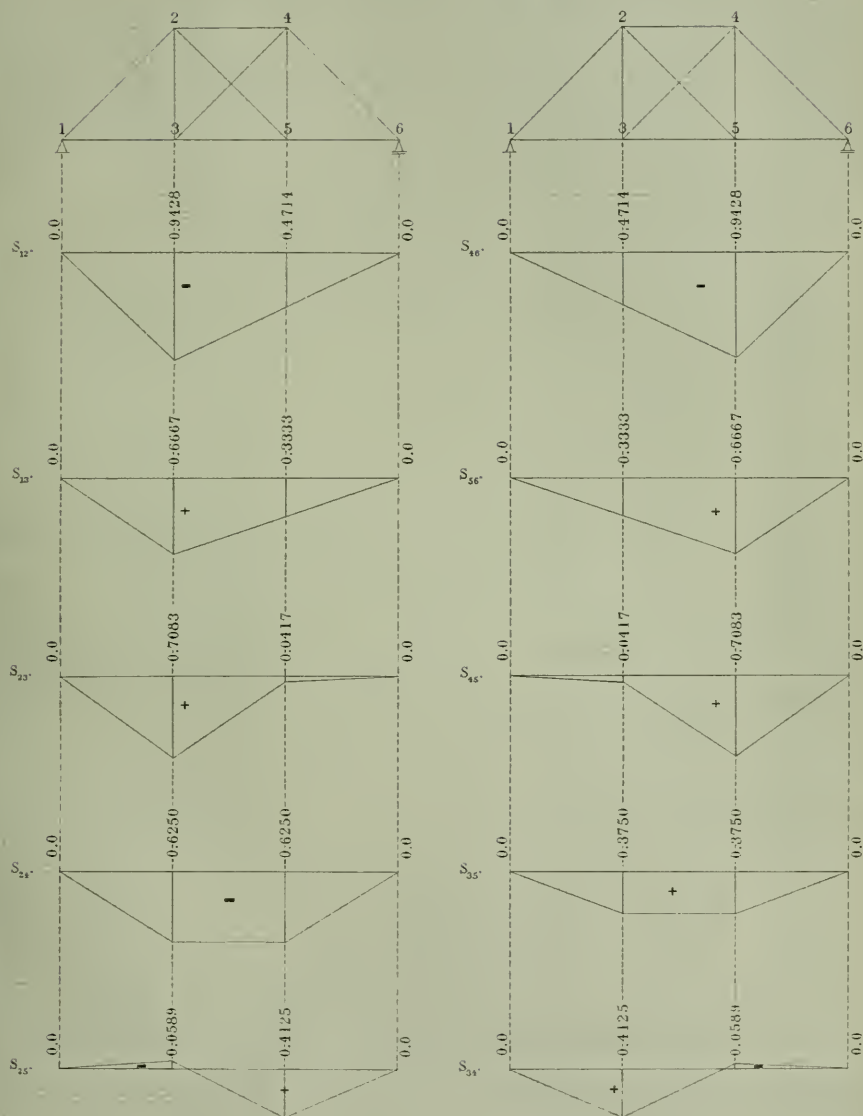
$$\begin{aligned} S_{13} &= L_{13}(p_3 - p_1) + M_{13}(q_3 - q_1) \\ &= (p_3)(10^5), \\ S_{23} &= L_{23}(p_3 - p_2) + M_{23}(q_3 - q_2) \\ &= (q_3 - q_2)(10^5), \\ S_{24} &= L_{24}(p_4 - p_2) + M_{24}(q_4 - q_2) \\ &= (p_4 - p_2)(10^5), \\ S_{25} &= L_{25}(p_5 - p_2) + M_{25}(q_5 - q_2) \\ &= (p_5 - p_2 + q_5 - q_2)(10^5/\sqrt{2}), \\ S_{34} &= L_{34}(p_4 - p_3) + M_{34}(q_4 - q_3) \\ &= (p_4 - p_3 - q_4 + q_3)(10^5/\sqrt{2}), \\ S_{35} &= L_{35}(p_5 - p_3) + M_{35}(q_5 - q_3) \\ &= (p_5 - p_3)(10^5), \\ S_{45} &= L_{45}(p_5 - p_4) + M_{45}(q_5 - q_4) \\ &= (q_5 - q_4)(10^5), \end{aligned}$$

TABLE B.

$S_{12} =$	$0.471,405P_2 +$	$0P_3$	$+ 0.471,405P_4 +$	$0P_5$	$+ 0P_6$	$- 0.942,809Q_2 - 0.942,809Q_3 - 0.471,405Q_4 - 0.471,405Q_5,$
$S_{13} =$	$0.666,667P_2 + 1.000,000P_3 +$	$1.000,000P_4 + 1.000,000P_5 +$	$1.000,000P_6 + 0.666,667Q_2 + 0.666,667Q_3 + 0.333,333Q_4 + 0.333,333Q_5,$			
$S_{23} =$	$-0.166,667P_2 +$	$0P_3$	$- 0.291,667P_4 - 0.125,000P_5 - 0.125,000P_6 - 0.166,667Q_2 + 0.708,333Q_3 + 0.166,667Q_4 + 0.041,667Q_5,$			
$S_{34} =$	$-0.500,000P_2 +$	$0P_3$	$+ 0.375,000P_4 - 0.125,000P_5 - 0.125,000P_6 - 0.500,000Q_2 - 0.625,000Q_3 - 0.500,000Q_4 - 0.625,000Q_5,$			
$S_{35} =$	$-0.235,702P_2 +$	$0P_3$	$- 0.058,926P_4 + 0.176,777P_5 + 0.176,777P_6 - 0.235,702Q_2 - 0.058,926Q_3 + 0.235,702Q_4 + 0.412,479Q_5,$			
$S_{36} =$	$0.235,702P_2 +$	$0P_3$	$+ 0.412,479P_4 + 0.176,777P_5 + 0.176,777P_6 + 0.235,702Q_2 + 0.412,479Q_3 - 0.235,702Q_4 - 0.058,926Q_5,$			
$S_{35} =$	$0.500,000P_2 +$	$0P_3$	$+ 0.375,000P_4 + 0.875,000P_5 + 0.875,000P_6 + 0.500,000Q_2 + 0.375,000Q_3 + 0.500,000Q_4 + 0.375,000Q_5,$			
$S_{46} =$	$0.166,667P_2 +$	$0P_3$	$+ 0.041,667P_4 - 0.125,000P_5 - 0.125,000P_6 + 0.166,667Q_2 + 0.041,667Q_3 - 0.166,667Q_4 + 0.708,333Q_5,$			
$S_{46} =$	$-0.471,405P_2 +$	$0P_3$	$- 0.471,405P_4 +$	$0P_5$	$+ 0P_6$	$- 0.471,405Q_2 - 0.471,405Q_3 - 0.942,809Q_4 - 0.942,809Q_5,$
$S_{56} =$	$0.333,333P_2 +$	$0P_3$	$+ 0.333,333P_4 +$	$0P_5$	$+ 1.000,000P_6 + 0.333,333Q_2 + 0.333,333Q_3 + 0.666,667Q_4 + 0.666,667Q_5,$	

Fig. 5a.

Stress influence lines for the redundant frame.



$$\begin{aligned}
S'_{46} &= L_{46}(p_6 - p_4) + M_{46}(q_6 - q_4) \\
&= (p_6 - p_4 - q_4)(10^5/\sqrt{2}), \\
S'_{56} &= L_{56}(p_6 - p_5) + M_{56}(q_6 - q_5) \\
&= (p_6 - p_5)(10^5).
\end{aligned}$$

Thus the required general expressions of the stresses of all the members are found as given in Table B.

In the general solution of the stress of each member, the coefficients of the vertical loads Q 's of the loaded chord are equivalent to the influence values of the moving load for the corresponding member. Hence the stress influence lines of the members for the live load can be obtained from the general solution as shown in Fig. 5a.

By the general solutions we can easily determine the displacement components of the panel points and the stresses of the members due to any system of loading.

First, if $P_2=0$, $P_3=0$, $P_4=0$, $P_5=0$, $P_6=0$,

$$Q_2=424 \text{ kg.}, \quad Q_3=353 \text{ kg.},$$

$$Q_4=424 \text{ kg.}, \quad Q_5=353 \text{ kg.},$$

that is, the case of the dead load, we get the displacements of the panel points as follows:

$$\begin{array}{ll}
p_1=0, & q_1=0, \\
p_2=0.015,540 \text{ cm.}, & q_2=0.031,080 \text{ cm.}, \\
p_3=0.007,770 \text{ cm.}, & q_3=0.033,728 \text{ cm.}, \\
p_4=0.006,888 \text{ cm.}, & q_4=0.031,080 \text{ cm.}, \\
p_5=0.014,658 \text{ cm.}, & q_5=0.033,728 \text{ cm.}, \\
p_6=0.022,428 \text{ cm.}, & q_6=0;
\end{array}$$

and the stresses of the members are

$$S'_{12} = (\rho_2 - q_2)(10^5/1/2) = -777.1/\sqrt{2} = -1,098.8 \text{ kg.},$$

$$S'_{13} = (\rho_3)(10^5) = 777.0 \text{ kg.},$$

$$S'_{23} = (q_3 - q_2)(10^5) = 264.8 \text{ kg.},$$

$$S'_{24} = (\rho_4 - p_2)(10^5) = -865.3 \text{ kg.},$$

$$S'_{25} = (\rho_5 - p_2 + q_3 - q_2)(10^5/1/2) = 124.8 \text{ kg.},$$

$$S'_{34} = (\rho_4 - p_3 - q_4 + q_3)(10^5/1/2) = 124.8 \text{ kg.},$$

$$S'_{35} = (\rho_5 - p_3)(10^5) = 688.8 \text{ kg.},$$

$$S'_{45} = (q_5 - q_4)(10^5) = 264.8 \text{ kg.},$$

$$S'_{46} = (\rho_6 - p_4 - q_4)(10^5/1/2) = -1,098.8 \text{ kg.},$$

$$S'_{56} = (\rho_6 - p_5)(10^5) = 777.0 \text{ kg.},$$

which are the dead load stresses of the members.

Secondly, if $P_2=0$, $P_3=0$, $P_4=0$, $P_5=0$, $P_6=0$,

$$Q_2=0, \quad Q_3=6,000 \text{ kg.},$$

$$Q_4=0, \quad Q_5=3,000 \text{ kg.},$$

that is, a case of live loads, we obtain the elastic displacements as follows :

$$p_1=0, \quad q_1=0,$$

$$p_2=0.085,00 \text{ cm.}, \quad q_2=0.185,00 \text{ cm.},$$

$$p_3=0.050,00 \text{ cm.}, \quad q_3=0.228,75 \text{ cm.},$$

$$p_4=0.028,75 \text{ cm.}, \quad q_4=0.175,00 \text{ cm.},$$

$$p_5=0.083,75 \text{ cm.}, \quad q_5=0.198,75 \text{ cm.},$$

$$p_6=0.123,75 \text{ cm.}, \quad q_6=0 ;$$

and the stresses of the members due to the loading are

$$S_{12} = -7,071.1 \text{ kg.}, \quad S_{13} = 5,000.0 \text{ kg.},$$

$$S'_{23} = 4,375.0 \text{ kg.}, \quad S_{24} = -5,625.0 \text{ kg.},$$

$$S_{25} = 883.88 \text{ kg.}, \quad S'_{34} = 2,298.1 \text{ kg.},$$

$$S_{33} = 3,375.0 \text{ kg.}, \quad S_{45} = 2,375.0 \text{ kg.},$$

$$S_{46} = -5,656.9 \text{ kg.}, \quad S_{56} = 4,000.0 \text{ kg.}.$$

Now let us compare the above results with those of another solution of the redundant structure.

Since the given frame is a once internally indeterminate structure, we take one of the diagonals as the redundant member, reducing the frame to a simple truss as the principal system.

Let X_a represent the stress produced by the applied loads in the redundant member 3-4. Then the redundant force may be determined by the equation

$$X_a = \Sigma S_o \bar{S}_a \rho / (\rho_a + \Sigma \bar{S}_a^2 \rho),$$

where S_o = the stress in any member of the principal system due to the given loads,

$\bar{S}_a$ = the stress in this member of the principal frame when no load other than $X_a = -1$ is active,

ρ = the extensibility of this member of the principal system $= 1/A.E$,

ρ_a = the extensibility of the redundant member $= 1_{34}/A_{34}.E_{34}$.

The values of the stresses are found for the principal system as follows:

Member.	S_o .	$\bar{S}_a$.	$S_o \bar{S}_a$.	$\bar{S}_a^2$.
1-2	$-5,000\sqrt{2}$	0	0	0
1-3	5,000	0	0	0
2-3	6,000	$1/\sqrt{2}$	$6,000/\sqrt{2}$	$1/2$
2-4	-4,000	$1/\sqrt{2}$	$-4,000/\sqrt{2}$	$1/2$

Member.	S_o .	$\bar{S}_a$.	$S_o\bar{S}_a$.	$\bar{S}_a^2$.
2-5	$-1,000\sqrt{2}$	-1	$1,000\sqrt{2}$	1
3-5	5,000	$1/\sqrt{2}$	$5,000/\sqrt{2}$	1/2
4-5	4,000	$1/\sqrt{2}$	$4,000/\sqrt{2}$	1/2
4-6	$-4,000\sqrt{2}$	0	0	0
5-6	4,000	0	0	0

$$\Sigma S_o\bar{S}_a = 6,500\sqrt{2}. \quad \Sigma \bar{S}_a^2 = 3.$$

$$\text{Hence} \quad X_a = \Sigma S_o\bar{S}_a\rho / (\rho_a + \Sigma \bar{S}_a^2\rho),$$

since the extensibility is the same for every member, we have

$$\begin{aligned} X_a &= \Sigma S_o\bar{S}_a / (1 + \Sigma \bar{S}_a^2) \\ &= 6,500\sqrt{2}/4 = 2,298.1 \text{ kg.}, \end{aligned}$$

Then the stress in any member of the principal system is given in the form

$$S = S_o - \bar{S}_a X_a.$$

For the member 1-2

$$S_{12} = S_{o_{12}} - \bar{S}_{a_{12}} X_a,$$

or substituting the values given in the table we obtain

$$S_{12} = -5,000\sqrt{2} - (0)X_a = -7,071.1 \text{ kg.}$$

Similarly for the stresses of the other members we have

$$S_{13} = 5,000 - (0)X_a = 5,000.0 \text{ kg.},$$

$$S_{23} = 6,000 - (1/\sqrt{2})X_a = 4,375.0 \text{ kg.},$$

$$S_{24} = -4,000 - (1/\sqrt{2})X_a = -5,625.0 \text{ kg.},$$

$$S_{25} = -1,000/\sqrt{2} - (-1)X_a = 883.88 \text{ kg.},$$

$$S_{35} = 5,000 - (1/\sqrt{2})X_a = 3,375.0 \text{ kg.},$$

$$S_{45} = 4,000 - (1/\sqrt{2})X_a = 2,375.0 \text{ kg.},$$

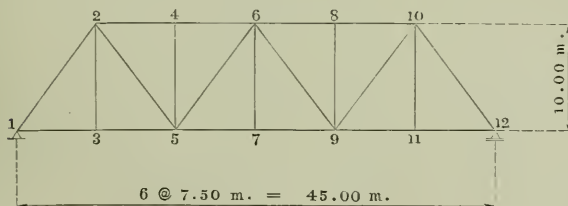
$$S_{46} = -4,000/\sqrt{2} - (0)X_a = -5,656.9 \text{ kg.},$$

$$S_{56} = 4,000 - (0)X_a = 4,000.0 \text{ kg.}.$$

These results coincide exactly with those of the previous solution.

Example 3. To determine the general solution of the elastic distortions of a through Warren bridge truss with suspenders shown in Fig. 6, assuming the modulus of elasticity $E=2,000,000$ kg. per sq. cm.

Fig. 6.



Given data are as follows:

Member.	Length l (cm.)	Area. A (cm. ²)	E.A/l. μ (kg./cm.)	Cos α .	Cos β .
1-2	1250	500	8.0×10^5	0.6	-0.8
1-3	750	300	8.0×10^5	1.0	0
2-3	1000	150	3.0×10^5	0	1.0

Member.	Length. l (cm.)	Area. A (cm. ²)	E.A/l. μ (kg./cm.)	$\cos\alpha$.	$\cos\beta$.
2-4	750	450	12.0×10^5	1.0	0
2-5	1250	300	4.8×10^5	0.6	0.8
3-5	750	300	8.0×10^5	1.0	0
4-5	1000	150	3.0×10^5	0	1.0
4-6	750	450	12.0×10^5	1.0	0
5-6	1250	200	3.2×10^5	0.6	-0.8
5-7	750	450	12.0×10^5	1.0	0
6-7	1000	150	3.0×10^5	0	1.0
6-8	750	450	12.0×10^5	1.0	0
6-9	1250	200	3.2×10^5	0.6	0.8
7-9	750	450	12.0×10^5	1.0	0
8-9	1000	150	3.0×10^5	0	1.0
8-10	750	450	12.0×10^5	1.0	0
9-10	1250	300	4.8×10^5	0.6	-0.8
9-11	750	300	8.0×10^5	1.0	0
10-11	1000	150	3.0×10^5	0	1.0
10-12	1250	500	8.0×10^5	0.6	0.8
11-12	750	300	8.0×10^5	1.0	0

Mem- ber.	L= μ . Cos α . (kg./cm.)	M= μ . Cos β . (kg./cm.)	L. Cos α . (kg./cm.)	L. Cos β . (kg./cm.)	M. Cos α . (kg./cm.)	M. Cos β . (kg./cm.)
1-2	4.80×10^5	-6.40×10^5	2.880×10^5	-3.840×10^5	-3.840×10^5	5.120×10^5
1-3	8.00×10^5	0	8.000×10^5	0	0	0
2-3	0	3.00×10^5	0	0	0	3.000×10^5
2-4	12.00×10^5	0	12.000×10^5	0	0	0
2-5	2.88×10^5	3.84×10^5	1.728×10^5	2.304×10^5	2.304×10^5	3.072×10^5
3-5	8.00×10^5	0	8.000×10^5	0	0	0
4-5	0	3.00×10^5	0	0	0	3.000×10^5
4-6	12.00×10^5	0	12.000×10^5	0	0	0
5-6	1.92×10^5	-2.56×10^5	1.152×10^5	-1.536×10^5	-1.536×10^5	2.048×10^5
5-7	12.00×10^5	0	12.000×10^5	0	0	0
6-7	0	3.00×10^5	0	0	0	3.000×10^5
6-8	12.00×10^5	0	12.000×10^5	0	0	0
6-9	1.92×10^5	2.56×10^5	1.152×10^5	1.536×10^5	1.536×10^5	2.048×10^5
7-9	12.00×10^5	0	12.000×10^5	0	0	0
8-9	0	3.00×10^5	0	0	0	3.000×10^5
8-10	12.00×10^5	0	12.000×10^5	0	0	0
9-10	2.88×10^5	-3.84×10^5	1.728×10^5	-2.304×10^5	-2.304×10^5	3.072×10^5
9-11	8.00×10^5	0	8.000×10^5	0	0	0
10-11	0	3.00×10^5	0	0	0	3.000×10^5
10-12	4.80×10^5	6.40×10^5	2.880×10^5	3.840×10^5	3.840×10^5	5.120×10^5
11-12	8.00×10^5	0	8.000×10^5	0	0	0

The general equations of equilibrium for any panel point (h) of a plane truss are

$$\Sigma X_h = 0, \text{ or } \Sigma_h [L_{hk} \{ (\rho_k - \rho_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} \}] = -P_h,$$

$$\Sigma Y_h = 0, \text{ or } \Sigma_h [M_{hk} \{ (\rho_k - \rho_h) \cos \alpha_{hk} + (q_k - q_h) \cos \beta_{hk} \}] = -Q_h,$$

where

$$L_{hk} = (P'_{hk} \cdot A_{hk} / l_{hk}) \cos \alpha_{hk},$$

$$M_{hk} = (P'_{hk} \cdot A_{hk} / l_{hk}) \cos \beta_{hk}.$$

The reaction conditions of the given framed structure are

$$p_1 = 0, \quad q_1 = 0, \quad \text{and} \quad q_{12} = 0.$$

Then apply the condition of $\Sigma X = 0$ to each panel point :
At the panel point (2)

$$\begin{aligned} \Sigma X_2 = 0, \text{ or } & L_{21} \{ (\rho_1 - \rho_2) \cos \alpha_{21} + (q_1 - q_2) \cos \beta_{21} \} \\ & + L_{23} \{ (\rho_3 - \rho_2) \cos \alpha_{23} + (q_3 - q_2) \cos \beta_{23} \} \\ & + L_{24} \{ (\rho_4 - \rho_2) \cos \alpha_{24} + (q_4 - q_2) \cos \beta_{24} \} \\ & + L_{25} \{ (\rho_5 - \rho_2) \cos \alpha_{25} + (q_5 - q_2) \cos \beta_{25} \} = -P_2, \end{aligned}$$

or inserting the values given in the table we have

$$-P_2(10^{-5}) = -16.608p_2 + 12.000p_4 + 1.728p_5 + 1.536q_2 + 2.304q_5.$$

Similarly we obtain at the panel points (3), (4), ..., (12),

$$-P_3(10^{-5}) = -16.000p_3 + 8.000p_5,$$

$$-P_4(10^{-5}) = 12.000p_2 - 24.000p_4 + 12.000p_6,$$

$$-P_5(10^{-5}) = 1.728p_2 + 8.000p_3 - 22.880p_5 + 1.152p_6 + 12.000p_7$$

$$+ 2.304q_2 - 0.768q_5 - 1.536q_6,$$

$$-P_6(10^{-5}) = 12.000p_4 + 1.152p_5 - 26.304p_6 + 12.000p_8 + 1.152p_9$$

$$- 1.536q_5 + 1.536q_9,$$

$$-P_7(10^{-5}) = 12.000p_5 - 24.000p_7 + 12.000p_9,$$

$$\begin{aligned}
-P_8(10^{-5}) &= 12.000p_6 - 24.000p_8 + 12.000p_{10}, \\
-P_9(10^{-5}) &= 1.152p_6 + 12.000p_7 - 22.880p_9 + 1.728p_{10} + 8.000p_{11} \\
&\quad + 1.536q_6 + 0.768q_9 - 2.304q_{10}, \\
-P_{10}(10^{-5}) &= 12.000p_6 + 1.728p_9 - 16.608p_{10} + 2.880p_{12} \\
&\quad - 2.304q_9 - 1.536q_{10}, \\
-P_{11}(10^{-5}) &= 8.000p_9 - 16.000p_{11} + 8.000p_{12}, \\
-P_{12}(10^{-5}) &= 2.880p_{10} + 8.000p_{11} - 10.880p_{12} + 3.840q_{10}.
\end{aligned}$$

Secondly apply the condition of $\Sigma Y=0$ to each panel point :

At the panel point (2)

$$\begin{aligned}
\Sigma Y_2=0, \text{ or } & M_{21}\{(\rho_1-p_2) \cos \alpha_{21} + (q_1-q_2) \cos \beta_{21}\} \\
& + M_{23}\{(p_3-p_2) \cos \alpha_{23} + (q_3-q_2) \cos \beta_{23}\} \\
& + M_{24}\{(\rho_4-p_2) \cos \alpha_{24} + (q_4-q_2) \cos \beta_{24}\} \\
& + M_{25}\{(\rho_5-p_2) \cos \alpha_{25} + (q_5-q_2) \cos \beta_{25}\} = -Q_2,
\end{aligned}$$

or inserting the values we have

$$-Q_2(10^{-5}) = 1.536p_2 + 2.304p_3 - 11.192q_2 + 3.000q_3 + 3.072q_5.$$

Similarly we obtain at the panel points (3), (4), ..., (11),

$$\begin{aligned}
-Q_3(10^{-5}) &= 3.000q_2 - 3.000q_3, \\
-Q_4(10^{-5}) &= -3.000q_4 + 3.000q_5, \\
-Q_5(10^{-5}) &= 2.304p_2 - 0.768p_5 - 1.536p_6 + 3.072q_2 + 3.000q_4 \\
&\quad - 8.120q_5 + 2.048q_6, \\
-Q_6(10^{-5}) &= -1.536p_5 + 1.536p_9 + 2.048q_5 - 7.096q_6 \\
&\quad + 3.000q_7 + 2.048q_9, \\
-Q_7(10^{-5}) &= 3.000q_6 - 3.000q_7, \\
-Q_8(10^{-5}) &= -3.000q_8 + 3.000q_9,
\end{aligned}$$

$$\begin{aligned}
-Q_9(10^{-6}) &= 1.536p_6 + 0.768p_9 - 2.304p_{10} + 2.048q_6 \\
&\quad + 3.000q_8 - 8.120q_9 + 3.072q_{10}, \\
-Q_{10}(10^{-6}) &= -2.304p_9 - 1.536p_{10} + 3.840p_{12} + 3.072q_9 \\
&\quad - 11.192q_{10} + 3.000q_{11}, \\
-Q_{11}(10^{-6}) &= 3.000q_{10} - 3.000q_{11}.
\end{aligned}$$

Thus for the equilibrium conditions of the given truss we have 21 equations as given in Table C.

Solving the linear simultaneous equations of ($p's$, $q's$) we obtain finally the general solutions of the elastic displacements of all the panel points in terms of the external forces ($P's$, $Q's$) as given in the following Table D.

In the general solution of the elastic displacements of the panel points, the coefficient of each vertical load Q of the loaded chord is equivalent to the influence value of the moving load for the corresponding panel point displacement. Hence the displacement influence lines of all the panel points can be obtained from the general solutions as shown in Figs. 6_a, 6_b, 6_c and 6_d.

Moreover the elastic displacement of any panel point h for any system of moving loads is expressed by

$$d_h = \sum Q \cdot \eta,$$

where d_h = required elastic displacement of any panel point h .

Q = vertical loads,

η = the ordinate of the displacement influence line of d_h
under its respective load Q .

C.

$$\begin{aligned}
& 0p_{11} + 0p_{12} + 1.536q_2 + 0q_3 + 0q_4 + 2.304q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 2.304q_2 + 0q_3 + 0q_4 - 0.768q_5 - 1.536q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 - 1.536q_5 + 0q_6 + 0q_7 + 0q_8 + 1.536q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 3.000p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 1.536q_6 + 0q_7 + 0q_8 + 0.768q_9 - 2.304q_{10} + 0q_{11}, \\
& 0p_{11} + 2.880p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 + 0q_8 - 2.304q_9 - 1.536q_{10} + 0q_{11}, \\
& 1.000p_{11} + 8.000p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 3.000p_{11} - 10.880p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 3.840q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} - 11.192q_2 + 3.000q_3 + 0q_4 + 3.072q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 3.000q_2 - 3.000q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 - 3.000q_4 + 3.000q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 3.072q_2 + 0q_3 + 3.000q_4 - 8.120q_5 + 2.048q_6 + 0q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 + 2.048q_5 - 7.096q_6 + 3.000q_7 + 0q_8 + 2.048q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 3.000q_6 - 3.000q_7 + 0q_8 + 0q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 - 3.000q_8 + 3.000q_9 + 0q_{10} + 0q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 2.048q_6 + 0q_7 + 3.000q_8 - 8.120q_9 + 3.072q_{10} + 0q_{11}, \\
& 0p_{11} + 3.840p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 + 0q_8 + 3.072q_9 - 11.192q_{10} + 3.000q_{11}, \\
& 0p_{11} + 0p_{12} + 0q_2 + 0q_3 + 0q_4 + 0q_5 + 0q_6 + 0q_7 + 0q_8 + 0q_9 + 3.000q_{10} - 3.000q_{11}
\end{aligned}$$

TABLE C.

$-P_2(10^{-5}) =$	$-16.608p_2 +$	$0p_3 +$	$12.000p_4 +$	$1.728p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$1.536q_2 +$	$0q_3 +$	$0q_4 +$	$2.304q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-P_3(10^{-5}) =$	$0p_2 -$	$16.000p_3 +$	$0p_4 +$	$8.000p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-P_4(10^{-5}) =$	$12.000p_2 +$	$0p_3 -$	$24.000p_4 +$	$0p_5 +$	$12.000p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-P_5(10^{-5}) =$	$1.728p_2 +$	$8.000p_3 +$	$0p_4 -$	$22.880p_5 +$	$1.152p_6 +$	$12.000p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$2.304q_2 +$	$0q_3 +$	$0q_4 -$	$0.768q_5 -$	$1.536q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-P_6(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$12.000p_4 +$	$1.152p_5 -$	$26.304p_6 +$	$0p_7 +$	$12.000p_8 +$	$1.152p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 -$	$1.536q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$1.536q_9 +$	$0q_{10} +$	$0q_{11},$
$-P_7(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$12.000p_5 +$	$0p_6 -$	$24.000p_7 +$	$0p_8 +$	$12.000p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-P_8(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$1.152p_6 +$	$12.000p_7 +$	$0p_8 -$	$22.880p_9 +$	$1.728p_{10} +$	$8.000p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$1.536q_6 +$	$0q_7 +$	$0q_8 +$	$0.768q_9 -$	$2.304q_{10} +$	$0q_{11},$
$-P_{10}(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$0p_6 +$	$0p_7 +$	$12.000p_8 +$	$1.728p_9 -$	$16.608p_{10} +$	$0p_{11} +$	$2.880p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 -$	$2.304q_9 -$	$1.536q_{10} +$	$0q_{11},$
$-P_{11}(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$8.000p_9 +$	$0p_{10} -$	$16.000p_{11} +$	$8.000p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-P_{12}(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$2.880p_{10} +$	$8.000p_{11} -$	$10.880p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$3.840q_{10} +$	$0q_{11},$
$-Q_2(10^{-5}) =$	$1.536p_2 +$	$0p_3 +$	$0p_4 +$	$2.304p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} -$	$11.192q_2 +$	$3.000q_3 +$	$0q_4 +$	$3.072q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-Q_3(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$3.000q_2 -$	$3.000q_3 +$	$0q_4 +$	$0q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-Q_4(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 -$	$3.000q_4 +$	$3.000q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-Q_5(10^{-5}) =$	$2.304p_2 +$	$0p_3 +$	$0p_4 -$	$0.768p_5 -$	$1.536p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$3.072q_2 +$	$0q_3 +$	$3.000q_4 -$	$8.120q_5 +$	$2.048q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-Q_6(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 -$	$1.536p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$1.536p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$2.048q_5 -$	$7.096q_6 +$	$3.000q_7 +$	$0q_8 +$	$2.048q_9 +$	$0q_{10} +$	$0q_{11},$
$-Q_7(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$3.000q_6 -$	$3.000q_7 +$	$0q_8 +$	$0q_9 +$	$0q_{10} +$	$0q_{11},$
$-Q_8(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$0q_6 +$	$0q_7 -$	$3.000q_8 +$	$3.000q_9 +$	$0q_{10} +$	$0q_{11},$
$-Q_9(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$1.536p_6 +$	$0p_7 +$	$0p_8 +$	$0.768p_9 -$	$2.304p_{10} +$	$0p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$2.048q_6 +$	$0q_7 +$	$3.000q_8 -$	$8.120q_9 +$	$3.072q_{10} +$	$0q_{11},$
$-Q_{10}(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 -$	$2.304p_9 -$	$1.536p_{10} +$	$0p_{11} +$	$3.840p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$3.072q_9 -$	$11.192q_{10} +$	$3.000q_{11},$
$-Q_{11}(10^{-5}) =$	$0p_2 +$	$0p_3 +$	$0p_4 +$	$0p_5 +$	$0p_6 +$	$0p_7 +$	$0p_8 +$	$0p_9 +$	$0p_{10} +$	$0p_{11} +$	$0p_{12} +$	$0q_2 +$	$0q_3 +$	$0q_4 +$	$0q_5 +$	$0q_6 +$	$0q_7 +$	$0q_8 +$	$0q_9 +$	$3.000q_{10} -$	$3.000q_{11}$

212 b

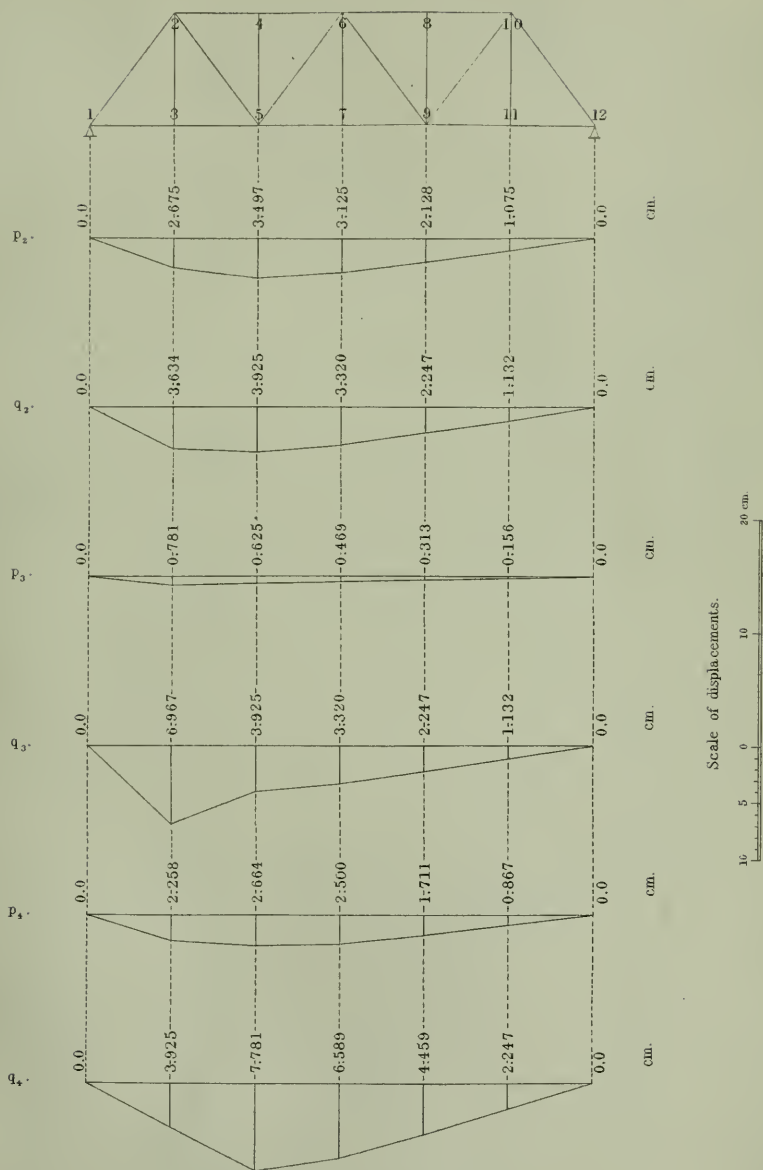
576Q ₃ + 3.497,299Q ₄ + 3.497,299Q ₅ + 3.125,000Q ₆ + 3.125,000Q ₇ + 2.127,701Q ₈ + 2.127,701Q ₉ + 1.075,424Q ₁₀ + 1.075,424Q ₁₁ ,
250Q ₃ + 0.625,000Q ₄ + 0.625,000Q ₅ + 0.468,750Q ₆ + 0.468,750Q ₇ + 0.312,500Q ₈ + 0.312,500Q ₉ + 0.156,250Q ₁₀ + 0.156,250Q ₁₁ ,
909Q ₃ + 2.663,966Q ₄ + 2.663,966Q ₅ + 2.500,000Q ₆ + 2.500,000Q ₇ + 1.711,034Q ₈ + 1.711,034Q ₉ + 0.867,091Q ₁₀ + 0.867,091Q ₁₁ ,
500Q ₃ + 1.250,000Q ₄ + 1.250,000Q ₅ + 0.937,500Q ₆ + 0.937,500Q ₇ + 0.625,000Q ₈ + 0.625,000Q ₉ + 0.312,500Q ₁₀ + 0.312,500Q ₁₁ ,
243Q ₃ + 1.830,633Q ₄ + 1.830,633Q ₅ + 1.875,000Q ₆ + 1.875,000Q ₇ + 1.294,367Q ₈ + 1.294,367Q ₉ + 0.658,758Q ₁₀ + 0.658,758Q ₁₁ ,
000Q ₃ + 1.875,000Q ₄ + 1.875,000Q ₅ + 1.875,000Q ₆ + 1.875,000Q ₇ + 1.250,000Q ₈ + 1.250,000Q ₉ + 0.625,000Q ₁₀ + 0.625,000Q ₁₁ ,
909Q ₃ + 1.413,966Q ₄ + 1.413,966Q ₅ + 1.250,000Q ₆ + 1.250,000Q ₇ + 0.461,034Q ₈ + 0.461,034Q ₉ + 0.242,091Q ₁₀ + 0.242,091Q ₁₁ ,
500Q ₃ + 2.500,000Q ₄ + 2.500,000Q ₅ + 2.812,500Q ₆ + 2.812,500Q ₇ + 1.875,000Q ₈ + 1.875,000Q ₉ + 0.937,500Q ₁₀ + 0.937,500Q ₁₁ ,
576Q ₃ + 0.997,299Q ₄ + 0.997,299Q ₅ + 0.625,000Q ₆ + 0.625,000Q ₇ - 0.372,299Q ₈ - 0.372,299Q ₉ - 0.174,576Q ₁₀ - 0.174,576Q ₁₁ ,
750Q ₃ + 2.812,500Q ₄ + 2.812,500Q ₅ + 3.281,250Q ₆ + 3.281,250Q ₇ + 2.500,000Q ₈ + 2.500,000Q ₉ + 1.718,750Q ₁₀ + 1.718,750Q ₁₁ ,
000Q ₃ + 3.125,000Q ₄ + 3.125,000Q ₅ + 3.750,000Q ₆ + 3.750,000Q ₇ + 3.125,000Q ₈ + 3.125,000Q ₉ + 2.500,000Q ₁₀ + 2.500,000Q ₁₁ ,
536Q ₃ + 3.925,058Q ₄ + 3.925,058Q ₅ + 3.320,313Q ₆ + 3.320,313Q ₇ + 2.246,817Q ₈ + 2.246,817Q ₉ + 1.132,089Q ₁₀ + 1.132,089Q ₁₁ ,
869Q ₃ + 3.925,058Q ₄ + 3.925,058Q ₅ + 3.320,313Q ₆ + 3.320,313Q ₇ + 2.246,817Q ₈ + 2.246,817Q ₉ + 1.132,089Q ₁₀ + 1.132,089Q ₁₁ ,
058Q ₃ + 11.114,005Q ₄ + 7.780,671Q ₅ + 6.588,542Q ₆ + 6.588,542Q ₇ + 4.458,912Q ₈ + 4.458,912Q ₉ + 2.246,817Q ₁₀ + 2.246,817Q ₁₁ ,
058Q ₃ + 7.780,671Q ₄ + 7.780,671Q ₅ + 6.588,542Q ₆ + 6.588,542Q ₇ + 4.458,912Q ₈ + 4.458,912Q ₉ + 2.246,817Q ₁₀ + 2.246,817Q ₁₁ ,
313Q ₃ + 6.588,542Q ₄ + 6.588,542Q ₅ + 9.733,073Q ₆ + 9.733,073Q ₇ + 6.588,542Q ₈ + 6.588,542Q ₉ + 3.320,313Q ₁₀ + 3.320,313Q ₁₁ ,
313Q ₃ + 6.588,542Q ₄ + 6.588,542Q ₅ + 9.733,073Q ₆ + 13.066,406Q ₇ + 6.588,542Q ₈ + 6.588,542Q ₉ + 3.320,313Q ₁₀ + 3.320,313Q ₁₁ ,
317Q ₃ + 4.458,912Q ₄ + 4.458,912Q ₅ + 6.588,542Q ₆ + 6.588,542Q ₇ + 11.114,005Q ₈ + 7.780,671Q ₉ + 3.925,058Q ₁₀ + 3.925,058Q ₁₁ ,
317Q ₃ + 4.458,912Q ₄ + 4.458,912Q ₅ + 6.588,542Q ₆ + 6.588,542Q ₇ + 7.780,671Q ₈ + 7.780,671Q ₉ + 3.925,058Q ₁₀ + 3.925,058Q ₁₁ ,
089Q ₃ + 2.246,817Q ₄ + 2.246,817Q ₅ + 3.320,313Q ₆ + 3.320,313Q ₇ + 3.925,058Q ₈ + 3.925,058Q ₉ + 3.633,536Q ₁₀ + 3.633,536Q ₁₁ ,
089Q ₃ + 2.246,817Q ₄ + 2.246,817Q ₅ + 3.320,313Q ₆ + 3.320,313Q ₇ + 3.925,058Q ₈ + 3.925,058Q ₉ + 3.633,536Q ₁₀ + 6.966,869Q ₁₁ .

TABLE D.

$$\begin{aligned}
p_2(10^6) &= 4.144,805P_2 + 1.041,667P_3 + 3.589,249P_4 + 2.083,333P_5 + 3.033,693P_6 + 2.500,000P_7 + 2.755,916P_8 + 2.916,667P_9 + 2.478,138P_{10} + 3.125,000P_{11} + 3.333,333P_{12} + 2.674,576Q_2 + 2.674,576Q_3 + 3.497,299Q_4 + 3.497,299Q_5 + 3.125,000Q_6 + 3.125,000Q_7 + 2.127,701Q_8 + 2.127,701Q_9 + 1.075,424Q_{10} + 1.075,424Q_{11}, \\
p_3(10^6) &= 1.041,667P_2 + 1.250,000P_3 + 1.041,667P_4 + 1.250,000P_5 + 1.041,667P_6 + 1.250,000P_7 + 1.041,667P_8 + 1.250,000P_9 + 1.041,667P_{10} + 1.250,000P_{11} + 1.250,000P_{12} + 0.781,250Q_2 + 0.781,250Q_3 + 0.625,000Q_4 + 0.625,000Q_5 + 0.468,750Q_6 + 0.468,750Q_7 + 0.312,500Q_8 + 0.312,500Q_9 + 0.156,250Q_{10} + 0.156,250Q_{11}, \\
p_4(10^7) &= 3.589,249P_2 + 1.041,667P_3 + 3.867,027P_4 + 2.083,333P_5 + 3.311,471P_6 + 2.500,000P_7 + 3.033,693P_8 + 2.916,667P_9 + 2.755,916P_{10} + 3.125,000P_{11} + 3.333,333P_{12} + 2.257,909Q_2 + 2.257,909Q_3 + 2.663,966Q_4 + 2.663,966Q_5 + 2.500,000Q_6 + 2.500,000Q_7 + 1.711,034Q_8 + 1.711,034Q_9 + 0.867,091Q_{10} + 0.867,091Q_{11}, \\
p_5(10^6) &= 2.083,333P_2 + 1.250,000P_3 + 2.083,333P_4 + 2.500,000P_5 + 2.083,333P_6 + 2.500,000P_7 + 2.083,333P_8 + 2.500,000P_9 + 2.083,333P_{10} + 2.500,000P_{11} + 2.500,000P_{12} + 1.562,500Q_2 + 1.562,500Q_3 + 1.250,000Q_4 + 1.250,000Q_5 + 0.937,500Q_6 + 0.937,500Q_7 + 0.625,000Q_8 + 0.625,000Q_9 + 0.312,500Q_{10} + 0.312,500Q_{11}, \\
p_6(10^6) &= 3.033,693P_2 + 1.041,667P_3 + 3.311,471P_4 + 2.083,333P_5 + 3.589,249P_6 + 2.500,000P_7 + 3.311,471P_8 + 2.916,667P_9 + 3.033,693P_{10} + 3.125,000P_{11} + 3.333,333P_{12} + 1.841,243Q_2 + 1.841,243Q_3 + 1.830,633Q_4 + 1.830,633Q_5 + 1.875,000Q_6 + 1.875,000Q_7 + 1.294,367Q_8 + 1.294,367Q_9 + 0.658,758Q_{10} + 0.658,758Q_{11}, \\
p_7(10^6) &= 2.500,000P_2 + 1.250,000P_3 + 2.500,000P_4 + 2.500,000P_5 + 2.500,000P_6 + 3.333,333P_7 + 2.500,000P_8 + 3.333,333P_9 + 2.500,000P_{10} + 3.333,333P_{11} + 3.333,333P_{12} + 1.875,000Q_2 + 1.875,000Q_3 + 1.875,000Q_4 + 1.875,000Q_5 + 1.875,000Q_6 + 1.875,000Q_7 + 1.250,000Q_8 + 1.250,000Q_9 + 0.625,000Q_{10} + 0.625,000Q_{11}, \\
p_8(10^6) &= 2.755,916P_2 + 1.041,667P_3 + 3.033,693P_4 + 2.083,333P_5 + 3.311,471P_6 + 2.500,000P_7 + 3.867,027P_8 + 2.916,667P_9 + 3.589,249P_{10} + 3.125,000P_{11} + 3.333,333P_{12} + 1.632,909Q_2 + 1.632,909Q_3 + 1.413,966Q_4 + 1.413,966Q_5 + 1.250,000Q_6 + 1.250,000Q_7 + 0.461,034Q_8 + 0.461,034Q_9 + 0.242,091Q_{10} + 0.242,091Q_{11}, \\
p_9(10^6) &= 2.916,667P_2 + 1.250,000P_3 + 2.916,667P_4 + 2.500,000P_5 + 2.916,667P_6 + 3.333,333P_7 + 2.916,667P_8 + 4.166,667P_9 + 2.916,667P_{10} + 4.166,667P_{11} + 4.166,667P_{12} + 2.187,500Q_2 + 2.187,500Q_3 + 2.500,000Q_4 + 2.500,000Q_5 + 2.812,500Q_6 + 2.812,500Q_7 + 1.875,000Q_8 + 1.875,000Q_9 + 0.937,500Q_{10} + 0.937,500Q_{11}, \\
p_{10}(10^6) &= 2.478,138P_2 + 1.041,667P_3 + 2.755,916P_4 + 2.083,333P_5 + 3.033,693P_6 + 2.500,000P_7 + 3.589,249P_8 + 2.916,667P_9 + 4.144,805P_{10} + 3.125,000P_{11} + 3.333,333P_{12} + 1.424,576Q_2 + 1.424,576Q_3 + 0.997,299Q_4 + 0.997,299Q_5 + 0.625,000Q_6 + 0.625,000Q_7 + 0.372,299Q_8 + 0.372,299Q_9 + 0.174,576Q_{10} + 0.174,576Q_{11}, \\
p_{11}(10^6) &= 3.125,000P_2 + 1.250,000P_3 + 3.125,000P_4 + 2.500,000P_5 + 3.125,000P_6 + 3.333,333P_7 + 3.125,000P_8 + 4.166,667P_9 + 3.125,000P_{10} + 5.416,667P_{11} + 5.416,667P_{12} + 2.343,750Q_2 + 2.343,750Q_3 + 2.812,500Q_4 + 2.812,500Q_5 + 3.281,250Q_6 + 3.281,250Q_7 + 2.500,000Q_8 + 2.500,000Q_9 + 1.718,750Q_{10} + 1.718,750Q_{11}, \\
p_{12}(10^6) &= 3.333,333P_2 + 1.250,000P_3 + 3.333,333P_4 + 2.500,000P_5 + 3.333,333P_6 + 3.333,333P_7 + 3.333,333P_8 + 4.166,667P_9 + 3.333,333P_{10} + 4.166,667P_{11} + 6.666,667P_{12} + 2.500,000Q_2 + 2.500,000Q_3 + 3.125,000Q_4 + 3.125,000Q_5 + 3.750,000Q_6 + 3.750,000Q_7 + 3.125,000Q_8 + 3.125,000Q_9 + 2.500,000Q_{10} + 2.500,000Q_{11}, \\
q_2(10^6) &= 2.674,576P_2 + 0.781,250P_3 + 2.257,909P_4 + 1.562,500P_5 + 1.841,243P_6 + 1.875,000P_7 + 1.632,909P_8 + 2.187,500P_9 + 1.424,576P_{10} + 2.343,750P_{11} + 2.500,000P_{12} + 3.633,536Q_2 + 3.633,536Q_3 + 3.925,058Q_4 + 3.925,058Q_5 + 3.320,313Q_6 + 3.320,313Q_7 + 2.246,817Q_8 + 2.246,817Q_9 + 1.132,089Q_{10} + 1.132,089Q_{11}, \\
q_3(10^6) &= 2.674,576P_2 + 0.781,250P_3 + 2.257,909P_4 + 1.562,500P_5 + 1.841,243P_6 + 1.875,000P_7 + 1.632,909P_8 + 2.187,500P_9 + 1.424,576P_{10} + 2.343,750P_{11} + 2.500,000P_{12} + 3.633,536Q_2 + 3.633,536Q_3 + 3.925,058Q_4 + 3.925,058Q_5 + 3.320,313Q_6 + 3.320,313Q_7 + 2.246,817Q_8 + 2.246,817Q_9 + 1.132,089Q_{10} + 1.132,089Q_{11}, \\
q_4(10^6) &= 3.497,299P_2 + 0.625,000P_3 + 2.663,966P_4 + 1.250,000P_5 + 1.830,633P_6 + 1.875,000P_7 + 1.413,966P_8 + 2.500,000P_9 + 0.997,299P_{10} + 2.812,500P_{11} + 3.125,000P_{12} + 3.925,058Q_2 + 3.925,058Q_3 + 11.114,005Q_4 + 7.780,671Q_5 + 6.588,542Q_6 + 6.588,542Q_7 + 4.458,912Q_8 + 4.458,912Q_9 + 2.246,817Q_{10} + 2.246,817Q_{11}, \\
q_5(10^6) &= 3.497,299P_2 + 0.625,000P_3 + 2.663,966P_4 + 1.250,000P_5 + 1.830,633P_6 + 1.875,000P_7 + 1.413,966P_8 + 2.500,000P_9 + 0.997,299P_{10} + 2.812,500P_{11} + 3.125,000P_{12} + 3.925,058Q_2 + 3.925,058Q_3 + 7.780,671Q_4 + 7.780,671Q_5 + 6.588,542Q_6 + 6.588,542Q_7 + 4.458,912Q_8 + 4.458,912Q_9 + 2.246,817Q_{10} + 2.246,817Q_{11}, \\
q_6(10^6) &= 3.125,000P_2 + 0.468,750P_3 + 2.500,000P_4 + 0.937,500P_5 + 1.875,000P_6 + 1.875,000P_7 + 1.250,000P_8 + 2.812,500P_9 + 0.625,000P_{10} + 3.281,250P_{11} + 3.750,000P_{12} + 3.320,313Q_2 + 3.320,313Q_3 + 6.588,542Q_4 + 6.588,542Q_5 + 9.733,073Q_6 + 9.733,073Q_7 + 6.588,542Q_8 + 6.588,542Q_9 + 3.320,313Q_{10} + 3.320,313Q_{11}, \\
q_7(10^6) &= 3.125,000P_2 + 0.468,750P_3 + 2.500,000P_4 + 0.937,500P_5 + 1.875,000P_6 + 1.875,000P_7 + 1.250,000P_8 + 2.812,500P_9 + 0.625,000P_{10} + 3.281,250P_{11} + 3.750,000P_{12} + 3.320,313Q_2 + 3.320,313Q_3 + 6.588,542Q_4 + 6.588,542Q_5 + 9.733,073Q_6 + 13.066,406Q_7 + 6.588,542Q_8 + 6.588,542Q_9 + 3.320,313Q_{10} + 3.320,313Q_{11}, \\
q_8(10^6) &= 2.127,701P_2 + 0.312,500P_3 + 1.711,034P_4 + 0.625,000P_5 + 1.294,367P_6 + 1.250,000P_7 + 0.461,034P_8 + 1.875,000P_9 + 0.372,299P_{10} + 2.500,000P_{11} + 3.125,000P_{12} + 2.246,817Q_2 + 2.246,817Q_3 + 4.458,912Q_4 + 4.458,912Q_5 + 6.588,542Q_6 + 6.588,542Q_7 + 11.114,005Q_8 + 7.780,671Q_9 + 3.925,058Q_{10} + 3.925,058Q_{11}, \\
q_9(10^6) &= 2.127,701P_2 + 0.312,500P_3 + 1.711,034P_4 + 0.625,000P_5 + 1.294,367P_6 + 1.250,000P_7 + 0.461,034P_8 + 1.875,000P_9 + 0.372,299P_{10} + 2.500,000P_{11} + 3.125,000P_{12} + 2.246,817Q_2 + 2.246,817Q_3 + 4.458,912Q_4 + 4.458,912Q_5 + 6.588,542Q_6 + 6.588,542Q_7 + 7.780,671Q_8 + 7.780,671Q_9 + 3.925,058Q_{10} + 3.925,058Q_{11}, \\
q_{10}(10^6) &= 1.075,424P_2 + 0.156,250P_3 + 0.867,091P_4 + 0.312,500P_5 + 0.658,758P_6 + 0.625,000P_7 + 0.242,091P_8 + 0.937,500P_9 + 0.174,576P_{10} + 1.718,750P_{11} + 2.500,000P_{12} + 1.132,089Q_2 + 1.132,089Q_3 + 2.246,817Q_4 + 2.246,817Q_5 + 3.320,313Q_6 + 3.320,313Q_7 + 3.925,058Q_8 + 3.925,058Q_9 + 3.633,536Q_{10} + 3.633,536Q_{11}, \\
q_{11}(10^6) &= 1.075,424P_2 + 0.156,250P_3 + 0.867,091P_4 + 0.312,500P_5 + 0.658,758P_6 + 0.625,000P_7 + 0.242,091P_8 + 0.937,500P_9 + 0.174,576P_{10} + 1.718,750P_{11} + 2.500,000P_{12} + 1.132,089Q_2 + 1.132,089Q_3 + 2.246,817Q_4 + 2.246,817Q_5 + 3.320,313Q_6 + 3.320,313Q_7 + 3.925,058Q_8 + 3.925,058Q_9 + 3.633,536Q_{10} + 6.966,869Q_{11}.
\end{aligned}$$

Fig. 6a.

Displacement influence lines for the Warren truss.

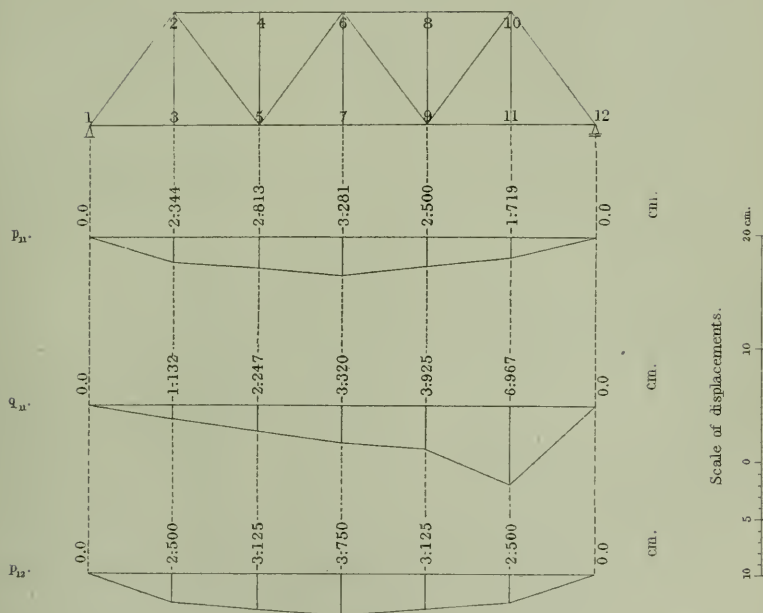


Influence ordinates 100,000 times actual.

212a

Fig. 6a.

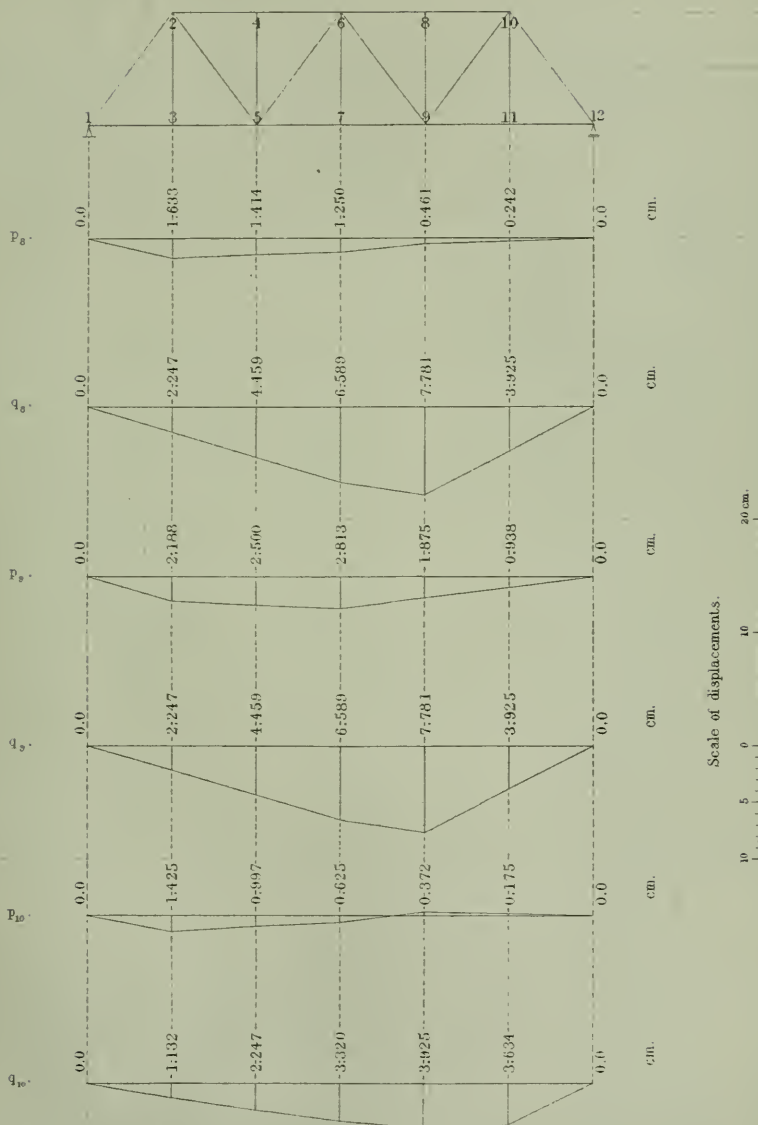
Displacement influence lines for the Warren truss.



Influence ordinates 100,000 times actual.

Fig. 6c.

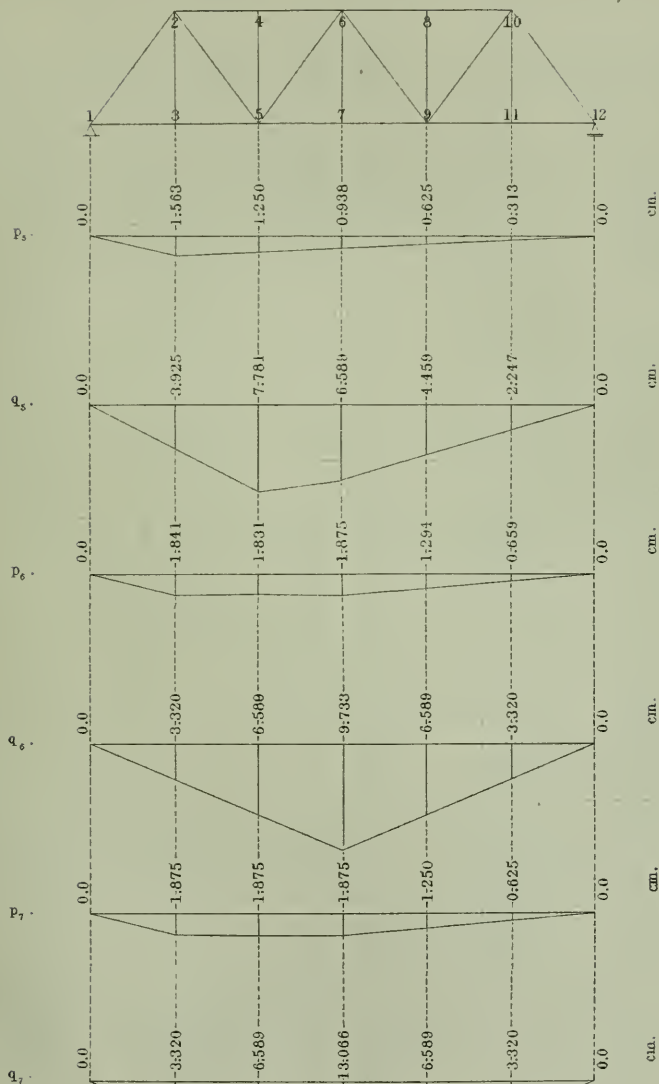
Displacement influence lines for the Warren truss.



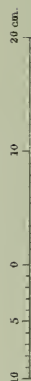
Influence ordinates 100,000 times actual.

Fig. 6_b.

Displacement influence lines for the Warren truss.



Scale of displacements.



Influence ordinates 100,000 times actual.

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Über die Differentialgleichung bezüglich der Airy'schen Spannungsfunktion und ihre Anwendung auf einige besondere Aufgaben in der Elastizitätslehre.

VON

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Über die Airy'sche Differentialgleichung.

1. Es handelt sich um die Differentialgleichung

$$(1) \quad \frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0,$$

worin F die Airy'sche Spannungsfunktion bezeichnet.

Diese Gleichung ist schon in manchen Lehrbüchern* dargestellt worden, trotzdem aber will ich hier kurz ihre Herleitung wiedergeben.

Es sei der ebene Spannungszustand eines Körpers vorausgesetzt. Bezeichnet man mit σ_x , σ_y und τ die wagerechten, lotrechten und tangentialen Komponenten der Spannung in einem beliebigen Punkt (x, y) des Körpers, so bestehen die Beziehungen

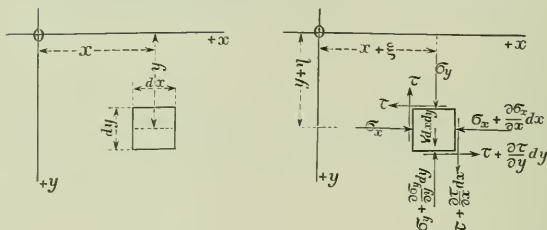
* Lorenz, Lehrbuch der Technischen Physik. 1913. Band IV. S. 497. Föppl, Vorlesungen über Technische Mechanik. 1907. V. S. 58.

$$(2) \quad \frac{\partial \sigma_x}{\partial x} = \frac{\partial \tau}{\partial y}$$

$$(3) \quad \frac{\partial \sigma_y}{\partial y} = \frac{\partial \tau}{\partial x} + \gamma,$$

wenn γ das Gewicht der Raumeinheit des Körpers angibt.

Fig. 1.



Differentiiert man die erste dieser Gleichungen nach x und die zweite nach y , so findet man

$$(4) \quad \frac{\partial^2 \sigma_x}{\partial x^2} = \frac{\partial^2 \sigma_y}{\partial y^2} = \frac{\partial^2 \tau}{\partial x \partial y}.$$

Hieraus geht hervor, dass σ_x und σ_y als Differentialquotienten einer und derselben Funktion von x und y angesehen werden können. Es zeigt sich nun weiterhin, dass diese Funktion gerade der im Eingang der Abhandlung genannten, mit F bezeichneten Funktion entspricht. Man erhält also

$$(5) \quad \sigma_x = \frac{\partial F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau = \frac{\partial^2 F}{\partial x \partial y}.$$

Stellt man mit ξ, η die Verschiebungen des Punktes (x, y) in den Richtungen der Koordinatenachsen X, Y dar, so ergibt sich unter Berücksichtigung der Querdehnungen

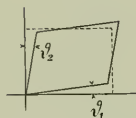
$$(6) \quad \begin{cases} \frac{\partial \xi}{\partial x} = \frac{1}{E} \left[\frac{\sigma_y}{m} - \sigma_x \right] \\ \frac{\partial \eta}{\partial y} = \frac{1}{E} \left[\frac{\sigma_x}{m} - \sigma_y \right], \end{cases}$$

wenn E die Elastizitätsziffer, m die Poissonzahl bedeuten.

Ferner, da

$$\begin{aligned} \vartheta &= \vartheta_1 + \vartheta_2 \\ &= \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \end{aligned}$$

Fig. 2.



ist, liefert die Beziehung $\vartheta = \frac{\tau}{G}$ die Gleichung

$$(7) \quad \tau = G \left[\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \right].$$

Bildet man aus Gln. (6), (7) je die Ausdrücke für $\frac{\partial^2 \eta}{\partial x \partial y}$ und setzt sie gleich, so erhält man die Gleichung

$$\frac{1}{E} \left[\frac{1}{m} \frac{\partial x_2}{\partial x} - \frac{\partial \sigma_y}{\partial x} \right] = \frac{1}{G} \frac{\partial \tau}{\partial y} - \frac{\partial^2 \xi}{\partial y^2},$$

welche durch nochmalige Differentiation nach x

$$\frac{\partial^2 \xi}{\partial y^2 \partial x} = \frac{1}{G} \frac{\partial^2 \tau}{\partial y \partial x} - \frac{1}{E} \left[\frac{1}{m} \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial x^2} \right]$$

liefert.

Führt man in diese Gleichung den Ausdruck $\frac{\partial^3 \xi}{\partial y^2 \partial x}$ aus Gl. (6) ein, so folgt:

$$\frac{1}{E} \left[\frac{1}{m} \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_x}{\partial y^2} \right] = \frac{1}{G} \frac{\partial^2 \tau}{\partial y \partial x} - \frac{1}{E} \left[\frac{1}{m} \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial x^2} \right].$$

Diese Gleichung geht wegen der Gl. (4) in

$$\frac{\partial^2 \sigma_x}{\partial y^2} = \left[\frac{2}{m} - \frac{E}{G} \right] \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_y}{\partial x^2}$$

über, die sich infolge der Beziehung

$$\frac{E}{G} = \frac{2(m+1)}{m}$$

weiterhin in

$$(8) \quad \frac{\partial^2 \sigma_z}{\partial y^2} = -2 \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_y}{\partial x^2}$$

vereinfacht.

Setzt man σ_z und σ_y aus Gl. (5) in der letzten Gleichung ein, so gelangt man zu der genannten Differentialgleichung (1).

2. Um diese Differentialgleichung zu integrieren, setzt man versuchsweise $F = e^{\beta x + \alpha y}$. Es ergibt sich dann

$$[\alpha^4 + 2\alpha^2\beta^2 + \beta^4]e^{\beta x + \alpha y} = 0$$

und folglich für α und β vier Bedingungsgleichungen je zwei von der Form

$$\beta = \pm i\alpha.$$

Daher entsprechen der Gl. (1) die folgenden vier partikulären Integralen:

$$(9) \quad \begin{cases} F_1 = e^{\alpha(y \pm ix)} \\ F_2 = ye^{\alpha(y \pm ix)} \end{cases}$$

Differentiiert man F_1 nach y zweimal, so ergibt sich

$$\frac{\partial^2 F_1}{\partial y^2} = -\alpha^2 e^{\alpha(y \pm ix)}.$$

Mithin folgt:

$$\sigma_z = \frac{\partial^2 F_1}{\partial y^2} = -\alpha^2 F_1.$$

Analog findet man

$$\sigma_z = \frac{\partial^2 F'_2}{\partial y^2} = a[\pm 2iF_1 - aF_2].$$

Man sieht also, dass σ_z sich als lineare Funktion der partikulären Integralen der Funktion F' angeben lässt. Daraus geht hervor, dass der Funktion σ_z auch eine Differentialgleichung von der Gestalt (1) entspricht, nämlich:

$$(10)_1 \quad \frac{\partial^4 \sigma_z}{\partial x^4} + 2 \frac{\partial^4 \sigma_z}{\partial x^2 \partial y^2} + \frac{\partial^4 \sigma_z}{\partial y^4} = 0.$$

Es ist von Wichtigkeit darauf hinzuweisen, dass die Differentialgleichung für die sogenannte Spannungsfunktion vollständig mit der für σ_z übereinstimmt. Dies kann man dadurch direkt nachweisen, dass man zwischen Gl. (5) und (8) σ_y eliminiert.

Auch für die Funktionen σ_y und τ bekommt man analog

$$(10)_2 \quad \begin{cases} \frac{\partial^4 \sigma_y}{\partial x^4} + 2 \frac{\partial^4 \sigma_y}{\partial x^2 \partial y^2} + \frac{\partial^4 \sigma_y}{\partial y^4} = 0 \\ \frac{\partial^4 \tau}{\partial x^4} + 2 \frac{\partial^4 \tau}{\partial x^2 \partial y^2} + \frac{\partial^4 \tau}{\partial y^4} = 0. \end{cases}$$

Bei der praktischen Anwendung der angegebenen Theorie wird man indessen finden, dass eine ganz allgemeine Anwendung der gefundenen Gleichungen mit grossen Schwierigkeiten verknüpft ist. Dies ist in erster Linie auf die Ungleichförmigkeit zurückzuführen, in der Gl. (3) wegen des Gliedes r sich befindet. Im folgenden wollen wir uns also auf die Fälle beschränken, in denen $r=0$ und ferner die vorkommenden Integralen mit Hilfe gewöhnlicher Formeln entwickelbar sind. Will man der Wirklichkeit des Spannungszustandes im Körper näher kommen und zwar indem man das Gewicht des Körpers berücksichtigt, so würde das am besten in der Weise geschehen, dass man den Einfluss

desselben gesondert berechnet und das Ergebnis zu dem unter der gemachten Annahme berechneten hinzufügt.

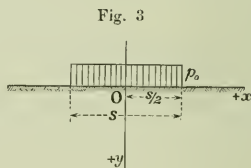
Aufgabe 1. Der Körper sei auf der einen Seite einer Ebene
unendlich ausgedehnt und über die Strecke s auf der
Ebene gleichförmig mit ρ_0 belastet.

1. Dieser Fall würde annähernd bei einem Erdreich zutreffen, das auf seiner Oberfläche Manergewicht trägt und zwar würde er der Wirklichkeit besonders dann nahe kommen, wenn man dabei noch annehmen würde, dass das Erdreich dem Hooke'schen Elastizitätsgesetze gehorcht.

Ersetzt man in Gl. (9) $e^{\pm iax}$ durch $\cos ax \pm i \sin ax$, so entstehen ohne weiteres die vier Lösungen für F

$$e^{ay} \cos ax \quad ye^{ay} \cos ax$$

$$e^{ay} \sin ax \quad ye^{ay} \sin ax.$$



Die Funktion σ_y , die sich als eine lineare Funktion von F darstellen lässt, besitzt auch für partikuläre Lösungen vier ähnliche Integralen. Da aber σ_y in bezug auf x eine gerade Funktion sein muss, müssen ihre Lösungen mit $\sin ax$ als Faktor ausgeschaltet werden. Ferner, weil σ_y bei wachsendem y beständig abnehmen muss, kommen nur die negativen Potenzen von e in Betracht. Das Gesagte gilt gerade für die Funktion F . Man erhält also für F den Ausdruck

$$F = [A_1 + A_2 y] e^{-ay} \cos ax,$$

wobei A_1 und A_2 zwei willkürliche Konstanten, α beliebige positive Zahl bezeichnen.

Da wir es mit einer homogenen linearen Gleichung (1) zu tun haben, ist auch

$$[A_1 + A_2 y] e^{-\alpha y} \cos \alpha x \cdot \psi'(a) da$$

ein Integral, wenn man unter $\psi'(a)$ eine beliebige Funktion von a und unter da das Differential von a versteht. Es werden aber auch Summen solcher für verschiedene a gebildeter Ausdrücke die Gleichung (1) befriedigen, und wir geben die gesuchte Funktion F als eine solche Summe an, nämlich als das folgende Integral:

$$F = \int_0^{\infty} [A_1 + A_2 y] e^{-\alpha y} \cos \alpha x \cdot \psi'(a) da.$$

Diese ist das allgemeine Integral der Gleichung (1). Die Merkwürdige Eigenschaft dieses Integrales, eine willkürlich bleibende Funktion zu enthalten, findet offenbar darin ihre analytische Erklärung, dass er hier unendlich viele von einander unabhängige partikuläre Integralen gibt.

Wir haben für die erste Grenzbedingung

$$\tau_{y=0} = 0.$$

Der Ausdruck für τ lässt sich gestalten durch die Beziehung $\tau = \frac{\partial^2 F}{\partial x \partial y}$. Da die Integrationsgrenze im letzten Ausdruck für F von x, y unabhängig sind, hat man demnach

$$\tau = \int_0^{\infty} a [A_1 + A_2 y - A_2] e^{-\alpha y} \sin \alpha x \cdot \psi'(a) da$$

und daher

$$\int_0^{\infty} a[A_1 - A_2] \sin ax \cdot \psi(a) da = 0.$$

Weil dies für eine beliebige Form von $\psi(a)$ gelten muss, ergibt sich

$$A_2 = aA_1.$$

Das allgemeine Integral nimmt dann die Form

$$F = \int_0^{\infty} [1 + ay] e^{-ay} \cos ax \cdot \psi(a) da$$

an, wenn man die Konstante A_1 mit der beliebigen Funktion $\psi(a)$ vereinigt.

Die zweite Grenzbedingung lautet :

$$[\sigma_y]_{y=0} = \left[\frac{\partial^2 F}{\partial x^2} \right]_{y=0} = f(x).$$

Hierbei ist $f(x)$ eine gegebene Funktion von x allein; es lautet

$$\text{Für } x \text{ zwischen } -\frac{s}{2} \text{ und } \frac{s}{2} \quad f(x) = p_0,$$

$$\text{„ die übrigen Werte von } x \quad f(x) = 0.$$

Man kann $\psi(a)$ so bestimmen, dass die Funktion σ_y für $y=0$ sich in $f(x)$ verwandelt. Man findet also

$$f(x) = - \int_0^{\infty} a^2 \cos ax \cdot \psi(a) da.$$

Die Funktion $f(x)$ kann man andererseits mittels des bekannten Fourier'schen Integrals ausdrücken, und dieses lautet, da $f(x)$ hierbei eine gerade Funktion ist,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \cos ax \cos at \, dt \, da.$$

Der Vergleich dieser beiden Darstellungen von $f(x)$ liefert

$$\psi(a) = -\frac{2}{a^2\pi} \int_0^\infty f(t) \cos at \, dt$$

und somit die gesuchte Funktion

$$F = -\frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{1}{a^2} f(t) \cos at (1+ay) e^{-zy} \cos ax \, dt \, da.$$

Wechselt man die Reihenfolge der Integrationen:

$$F = -\frac{2}{\pi} \int_0^\infty \frac{1+ay}{a^2} e^{-zy} \cos ax \, da \int_0^\infty f(t) \cos at \, dt,$$

so lässt sich die Integration nach t ausführen. Man erhält

$$\int_0^\infty f(t) \cos at \, dt = \int_0^{\frac{\pi}{2}} p_0 \cos at \, dt = p_0 \frac{\sin\left(\frac{as}{2}\right)}{a}.$$

Wir gewinnen also für F

$$F = -\frac{2p_0}{\pi} \int_0^\infty \frac{[1+ay] e^{-zy} \cos ax \sin\left(\frac{as}{2}\right)}{a^3} da.$$

Die Integration ist ziemlich umständlich. Anstatt die Funktion F unmittelbar zu bestimmen, wollen wir uns also dazu der Gleichung bedienen, um den Ausdruck für σ_y daraus abzuleiten. Dies geschieht folgendermassen: Man differenziert den letzten Ausdruck für F zweimal nach x , und findet infolge der Beziehung $\sigma_y = -\frac{\partial^2 F}{\partial x^2}$

$$\begin{aligned} \sigma_y &= \frac{2p_0}{\pi} \int_0^\infty [1+ay] e^{-zy} \cos ax \frac{\sin\left(\frac{as}{2}\right)}{a} da \\ &= \frac{p_0}{\pi} \left[\int_0^\infty e^{-zy} \left[\sin\left\{a\left(\frac{s}{2}+x\right)\right\} + \sin\left\{a\left(\frac{s}{2}-x\right)\right\} \right] da + y \int_0^\infty e^{-zy} \left[\sin\left\{a\left(\frac{s}{2}-x\right)\right\} \right] da \right] \end{aligned}$$

$$+ \sin \left\{ a \left(\frac{s}{2} - x \right) \right\} \Big] da \Big].$$

Die Integration lässt sich vermittle der Formeln

$$\begin{cases} \int_0^x \frac{e^{-ax} \sin bx}{x} dx = \arctg \left[\frac{b}{a} \right] & (\text{für } a > 0) \\ \int_0^x e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2} & ,, \end{cases}$$

durchführen. Man findet

$$(11) \quad \sigma_y = \frac{\rho_0}{\pi} \left[\arctg \left(\frac{\frac{s}{2} + x}{y} \right) + \arctg \left(\frac{\frac{s}{2} - x}{y} \right) \right. \\ \left. + y \left\{ \frac{\frac{s}{2} + x}{y^2 + \left(\frac{s}{2} + x \right)^2} + \frac{\frac{s}{2} - x}{y^2 + \left(\frac{s}{2} - x \right)^2} \right\} \right].$$

2. Es ist jetzt τ , σ_x zu bestimmen. Differenziert man Gl. (11) nach y , so folgt:

$$\frac{\partial \sigma_y}{\partial y} = \frac{-2\rho_0 y^2}{\pi} \left[\frac{\frac{s}{2} + x}{\left\{ y^2 + \left(\frac{s}{2} + x \right)^2 \right\}^2} + \frac{\frac{s}{2} - x}{\left\{ y^2 + \left(\frac{s}{2} - x \right)^2 \right\}^2} \right].$$

Unter Berücksichtigung der Beziehung $\tau = \int \frac{\partial \sigma_y}{\partial y} dx$, in der τ gleichzeitig mit y verschwinden muss, erhält man

$$(12) \quad \tau = \frac{\rho_0 y^2}{\pi} \left[\frac{1}{y^2 + \left(\frac{s}{2} + x \right)^2} - \frac{1}{y^2 + \left(\frac{s}{2} - x \right)^2} \right].$$

Die Differentiation der Gl. (12) nach y liefert

$$\frac{\partial \tau}{\partial y} = \frac{2\rho_0 y}{\pi} \left[\frac{\left(\frac{s}{2} + x\right)^2}{\left\{y^2 + \left(\frac{s}{2} + x\right)^2\right\}^2} - \frac{\left(\frac{s}{2} - x\right)^2}{\left\{y^2 + \left(\frac{s}{2} - x\right)^2\right\}^2} \right]$$

und folglich, da $\sigma_x = \int \frac{\partial \tau}{\partial y} dx$, in welchem Ausdruck σ_x für $y=0$ und $x > \frac{s}{2}$ verschwinden muss,

$$(13) \quad \sigma_x = \frac{\rho_0}{\pi} \left[\operatorname{arctg} \left(\frac{\frac{s}{2} + x}{y} \right) \operatorname{arctg} \left(\frac{\frac{s}{2} - x}{y} \right) - y \left\{ \frac{\frac{s}{2} + x}{y^2 + \left(\frac{s}{2} + x\right)^2} + \frac{\frac{s}{2} - x}{y^2 + \left(\frac{s}{2} - x\right)^2} \right\} \right].$$

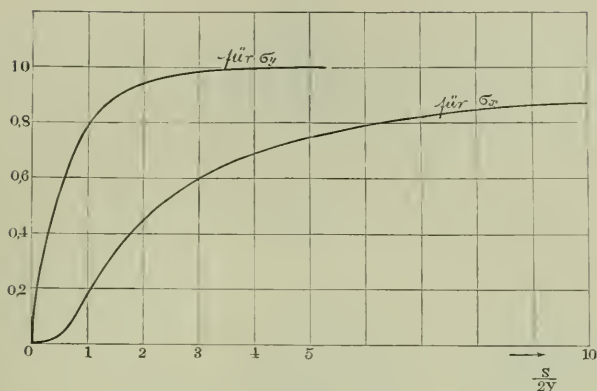
3. Setzt man in den Ausdrücken für σ_y , σ_x und τ $x=0$, so erhält man für die Funktionswerte auf der y -Achse

$$(14) \quad \begin{cases} \sigma_y = \frac{2\rho_0}{\pi} \left[\operatorname{arctg} \left(\frac{s}{2y} \right) + \frac{s}{2y \left\{ 1 + \left(\frac{s}{2y} \right)^2 \right\}} \right] \\ \sigma_x = \frac{2\rho_0}{\pi} \left[\operatorname{arctg} \left(\frac{s}{2y} \right) - \frac{s}{2y \left\{ 1 + \left(\frac{s}{2y} \right)^2 \right\}} \right] \\ \tau = 0. \end{cases}$$

Die Ausdrücke in Klammern und also die Koeffizienten von ρ_0 hängen nur vom Verhältnis $\frac{s}{2y}$ ab, und werden folgendermassen angegeben werden. Fig. 4 stellt die Ergebnisse graphisch dar.

$\frac{s}{2y}$	$(I) = \arctg\left(\frac{s}{2y}\right)$	$(II) = \frac{s}{2y} \left\{ 1 + \left(\frac{s}{2y} \right)^2 \right\}$	$(I) + (II)$	$\frac{2}{\pi} [(I) + (II)]$	$(I) - (II)$	$\frac{2}{\pi} [(I) - (II)]$
0	0	0	0	0	0	0
0,1	0,0966 87	0,0900 99	0,19867 86	0,12648 3	0,00065 88	0,00041 9
0,2	0,19739 56	0,19230 77	0,38970 33	0,24769 2	0,00508 79	0,00323 9
0,3	0,29145 68	0,27522 94	0,56668 62	0,36076 4	0,01622 74	0,01033 1
0,4	0,38050 63	0,34482 76	0,72533 39	0,46176 2	0,03567 87	0,02271 4
0,5	0,46364 76	0,40000 00	0,86364 76	0,55981 5	0,06364 76	0,04051 9
0,6	0,54041 95	0,44117 65	0,98159 60	0,62490 3	0,09924 30	0,06318 0
0,7	0,60490 82	0,46979 87	1,07470 69	0,68418 0	0,13510 95	0,08601 3
0,8	0,67474 10	0,48780 49	1,16254 59	0,74010 0	0,18693 61	0,11900 7
0,9	0,73281 51	0,49723 76	1,23005 27	0,78307 6	0,23557 75	0,14997 3
1,0	0,78539 82	0,50000 00	1,28539 82	0,81831 0	0,28539 82	0,18169 0
1,5	0,98279 37	0,46153 85	1,44433 22	0,91949 0	0,52125 52	0,33184 1
2,0	1,10714 87	0,40000 00	1,50714 87	0,95948 1	0,70714 87	0,45018 5
2,5	1,19029 00	0,34482 76	1,53511 76	0,97728 6	0,84546 24	0,53823 8
3,0	1,24904 58	0,30000 00	1,54904 58	0,98615 3	0,94904 58	0,60418 1
4,0	1,32581 77	0,23529 41	1,56111 18	0,99383 5	1,09052 36	0,69424 9
5,0	1,37340 08	0,19230 77	1,56570 85	0,99676 1	1,18109 31	0,75190 7
6,0	1,40564 76	0,16216 22	1,56780 98	0,99809 9	1,24348 54	0,77162 7
7,0	1,42899 93	0,14000 00	1,56899 93	0,99885 6	1,28899 93	0,82060 2
8,0	1,44644 14	0,12307 69	1,56951 83	0,99918 8	1,32336 45	0,84248 0
9,0	1,46013 91	0,10975 61	1,56989 52	0,99942 6	1,35038 30	0,85968 1
10,0	1,47112 77	0,09900 99	1,57013 76	0,99958 1	1,37211 78	0,87351 7

Fig. 4.



Zur näheren Erklärung nehmen wir ein Zahlenbeispiel an, in dem $s=600$ cm. Die Berechnungsergebnisse sind auf Fig. 5, 6, 7 graphisch dargestellt.

Die Hauptspannungen und ihre Richtung berechnet man nach den bekannten Formeln:

$$(15) \quad \begin{cases} \sigma_1 \text{ und } \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2} \sqrt{4\tau^2 + (\sigma_x - \sigma_y)^2} \\ \operatorname{tg} 2\delta = \frac{-2\tau}{\sigma_x - \sigma_y}. \end{cases}$$

Fig. 8 bezeichnet den Verlauf der Hauptspannungen für den soeben behandelten Zahlenbeispiel.

Fig. 5.

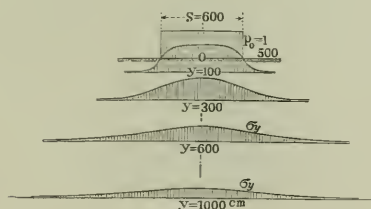


Fig. 6.

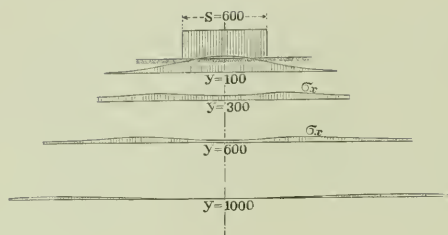


Fig. 7.

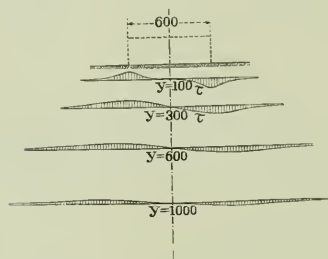
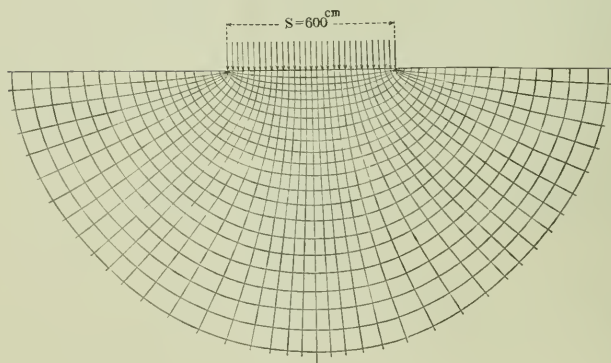


Fig. 8.



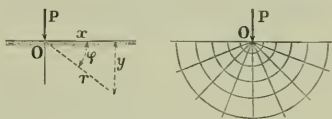
4. Will man die oben erwähnte Theorie auf einen Fall ausdehnen, in dem die gleichförmig verteilte Last p_0 auf der Strecke s zu einer Einzellast P konzentriert ist, so geht man von der Erwägung aus, dass die Grösse s unendlich klein wird, indem das Produkt $p_0 s$ stets der gegebenen Last P gleich bleibt, nämlich:

$$p_0 = \frac{P}{s}.$$

Demgemäss ergibt sich

$$\begin{aligned} \int_0^\infty f(t) \cos at \, dt &= \int_0^{\frac{s}{a}} \frac{P}{s} \cos at \, dt \\ &= \frac{P}{a} \frac{\sin\left(\frac{as}{2}\right)}{s}. \end{aligned}$$

Fig. 9.



Da

$$\lim_{s \rightarrow 0} \left[\frac{\sin\left(\frac{as}{2}\right)}{s} \right] = \frac{a}{2},$$

folgt:

$$F = \frac{-P}{\pi} \int_0^\infty \frac{[1 + dy] e^{-ay} \cos ax}{a^2} da.$$

Daraus erhält man ohne weiteres

$$\begin{aligned} \sigma_y &= \frac{\partial^2 F}{\partial x^2} = \frac{P}{\pi} \int_0^\infty [1 + ay] e^{-ay} \cos ax \, da \\ (16) \quad &= \frac{2P}{\pi} \frac{y^3}{(y^2 + x^2)^3}, \end{aligned}$$

einen Ausdruck, der, wenn $y=0$, für alle Werte von x , mit Ausnahme von $x=0$, gleich Null wird. Für den ausgenommenen Fall, dem der Punkt 0 entspricht, ist er unendlich gross.

Ferner nach Gl. (5) folgt :

$$(17) \quad \begin{cases} \sigma_x = \frac{P}{\pi} \int_0^\infty [1 - ay] e^{-xy} \cos ax dx = \frac{2P}{\pi} \frac{x^2 y}{(y^2 + x^2)^2} \\ \tau = \frac{-Py}{\pi} \int_0^\infty a e^{-xy} \sin ax da = -\frac{2P}{\pi} \frac{xy^2}{(y^2 + x^2)^2}. \end{cases}$$

Man erkennt daraus, dass σ_x und τ , falls entweder $x=0$ oder $y=0$ ist, also überall auf den beiden Koordinatenachsen verschwinden.

$$\text{Setzt man } r = \sqrt{x^2 + y^2}, \quad \frac{y}{r} = \sin \varphi, \quad \frac{x}{r} = \cos \varphi,$$

so erhält man

$$(18) \quad \begin{cases} \sigma_y = \frac{2Py}{\pi r^2} \sin^2 \varphi \\ \sigma_x = \frac{2Py}{\pi r^2} \cos^2 \varphi \\ \tau = \frac{-2Px}{\pi r^2} \sin^2 \varphi \end{cases}$$

und somit

$$tg 2\varphi = tg \varphi$$

oder

$$\varphi = \varphi.$$

Die Hauptspannungslinien lassen sich also als vom Lastpunkte aus ausgehende strahlenförmige Linien gestalten, die von konzentrischen Kreisen senkrecht durchschnitten werden.

Aufgabe 2. Der Körper sei trapezförmig belastet.

Wir wollen diesmal von einer Gleichung der Form (10) ausgehen, die die Spannung selbst z.B. σ_y als unbekannte Funktion enthält.

Es handelt sich hierbei um eine ungerade Funktion. Man nimmt also die partikulären Integralen

$$e^{-\alpha y} \cos\{a(t-x)\}$$

$$ye^{-\alpha y} \cos\{a(t-x)\}$$

an und versucht ob mit diesen eine eindeutige Auflösung der Aufgabe möglich sei.

Wie im vorigen Fall ergibt sich also das allgemeine Integral der Funktion σ_y

$$\sigma_y = \int_0^{\infty} [A_1 + A_2 y] e^{-\alpha y} \cos\{a(t-x)\} \cdot \psi(a) da$$

und ferner unter Berücksichtigung der Bedingung

$$[\tau]_{y=0} = \left[\int \frac{\partial \sigma_y}{\partial y} dx \right]_{y=0} = 0 \text{ wieder}$$

$$A_2 = \alpha A_1.$$

Es lautet somit

$$\sigma_y = \int_0^{\infty} [1 + \alpha y] e^{-\alpha y} \cos\{a(t-x)\} \cdot \psi(a) da.$$

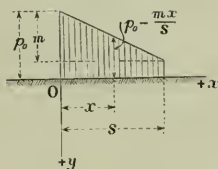
Es unterliegt also σ_y derselben Funktionsform wie vorher F , und wir sind in der Lage σ_y genau so zu bestimmen, wie früher F , wenn die den Grenzzustand charakterisierende Funktion $f(x)$ analog befriedigt wird. Es lautet hierbei:

$$\text{Für } x \text{ zwischen } 0 \text{ und } s, \quad f(x) = p_0 - \frac{mx}{s},$$

„ die übrigen Werte von x , $f(x) = 0$.

Setzt man im obigen Ausdruck für σ_y $y=0$, so ergibt sich

Fig. 10.



$$f(x) = [\sigma_y]_{y=0} = \int_0^\infty \cos\{a(t-x)\} \cdot \zeta(a) da.$$

Da andererseits

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-x}^x f(t) \cos\{a(t-x)\} dt da,$$

erhält man

$$\zeta(a) = \frac{1}{\pi} \int_{-x}^\infty f(t) dt$$

und daher ist

$$\sigma_y = \frac{1}{\pi} \int_0^\infty \int_{-x}^\infty f(t) \cdot (1+ay) e^{-ay} \cos a(t-x) dt da.$$

Führt man die Integration nach t , aus so bekommt man

$$\begin{aligned} \int_{-x}^\infty f(t) \cdot \cos a(t-x) dt &= \int_0^\infty \left[\rho_0 - \frac{mt}{s} \right] \cos a(t-x) dt \\ &= \rho_0 \left[\frac{\sin a(s-x) + \sin ax}{a} \right] - \frac{m}{s} \left[\frac{\cos a(s-x) - \cos ax}{a^2} \right] \\ &\quad - \frac{m}{s} \left[\frac{(s-x) \sin a(s-x) - x \sin ax}{a} \right] - \frac{mx}{s} \left[\frac{\sin a(s-x) + \sin ax}{a} \right]. \end{aligned}$$

Wir wollen uns nun mit einigen bestimmten Integralen beschäftigen, auf die wir bei weiterer Integration stossen. Wir haben Zunächst die Form

$$\int_0^\infty \frac{e^{-ax} [\cos b_1 x - \cos b_2 x]}{x} dx, \quad (a > 0)$$

Wir setzen

$$\varphi(k) = \int_0^\infty \frac{e^{-(k+a_1)x} \cos b_1 x - e^{-(k+a_2)x} \cos b_2 x}{x} dx,$$

in dem a_1, a_2 positiv ist. Man findet

$$\frac{d\varphi}{dk} = \int_0^\infty e^{-(k+a_2)x} \cos b_2 x dx - \int_0^\infty e^{-(k+a_1)x} \cos b_1 x dx$$

$$\begin{aligned}
&= \frac{k+a_2}{[k+a_2]^2+b_2^2} - \frac{k+a_1}{[k+a_1]^2+b_1^2} \\
&= \frac{1}{2} \frac{d}{dk} [\ln\{(k+a_2)^2+b_2^2\} - \ln\{(k+a_1)^2+b_1^2\}] \\
&= \frac{1}{2} \frac{d}{dk} \ln \left[\frac{(k+a_2)^2+b_2^2}{(k+a_1)^2+b_1^2} \right].
\end{aligned}$$

Da für $k=\infty$ die Funktion $\varphi(k)$ verschwindet, folgt:

$$\varphi(x) = \frac{1}{2} \ln \left[\frac{(k+a_2)^2+b_2^2}{(k+a_1)^2+b_1^2} \right].$$

Setzt man darin $k=0$, $a_1=a_2=a$, so erhält man

$$\int_0^\infty \frac{e^{-ax} [\cos b_1 x - \cos b_2 x]}{x} dx = \frac{1}{2} \ln \left[\frac{a^2+b_2^2}{a^2+b_1^2} \right].$$

Ferner bekommen wir die Form

$$\varphi(a) = \int_0^\infty \frac{e^{-ax} [\cos b_1 x - \cos b_2 x]}{x^2} dx.$$

Durch Differentiation nach a ergibt sich

$$\begin{aligned}
\frac{d\varphi}{da} &= - \int_0^\infty \frac{e^{-ax} [\cos b_1 x - \cos b_2 x]}{x} dx \\
&= - \frac{1}{2} \ln \left[\frac{a^2+b_2^2}{a^2+b_1^2} \right].
\end{aligned}$$

Daher folgt:

$$\varphi(a) = \frac{1}{2} \left[\int \ln(a^2+b_1^2) da - \int \ln(a^2+b_2^2) da \right],$$

welches mit Beachtung der Formel

$$\int \ln(x^2+y^2) dx = -2x + x \ln(x^2+y^2) + 2y \operatorname{arctg}\left(\frac{x}{y}\right)$$

sich in

$$\varphi(a) = \frac{1}{2} \left[a \ln \left\{ \frac{a^2+b_1^2}{a^2+b_2^2} \right\} + 2b_1 \operatorname{arctg}\left(\frac{a}{b_1}\right) - 2b_2 \operatorname{arctg}\left(\frac{a}{b_2}\right) \right] + C$$

entwickeln lässt. Da für $a=0$ das Integral in

$$\begin{aligned}\varphi(a) &= \int_0^{\infty} \frac{\cos b_1 x - \cos b_2 x}{x^2} dx \\ &= \frac{(b_2 - b_1)\pi}{2}\end{aligned}$$

übergeht, nimmt die Konstante C den Wert $\frac{(b_2 - b_1)\pi}{2}$ an.

Es gilt also

$$\begin{aligned}\int_0^{\infty} \frac{e^{-ax} [\cos b_1 x - \cos b_2 x]}{x^2} dx &= \frac{a}{2} \ln \left\{ \frac{a^2 + b_1^2}{a^2 + b_2^2} \right\} + b_1 \arctg \left(\frac{a}{b_1} \right) \\ &\quad - b_2 \arctg \left(\frac{a}{b_2} \right) + \frac{(b_2 - b_1)\pi}{2}.\end{aligned}$$

Die Integration der Funktion σ_y kann nun vollzogen werden. Das Ergebnis lautet

$$\begin{aligned}(19) \quad \sigma_y &= \frac{1}{\pi} \left[p_0 - \frac{mx}{s} \right] \left[\arctg \left(\frac{s-x}{y} \right) + \arctg \left(\frac{x}{y} \right) + y \left\{ \frac{s-x}{y^2 + (s-x)^2} \right. \right. \\ &\quad \left. \left. + \frac{x}{y^2 + x^2} \right\} \right] - \frac{my}{s\pi} \left[\frac{(s-x)^2}{y^2 + (s-x)^2} - \frac{x^2}{y^2 + x^2} \right].\end{aligned}$$

Differenziert man die letzte Funktion nach y , so folgt:

$$\begin{aligned}\frac{\partial \sigma_y}{\partial y} &= \frac{-2y^2}{\pi} \left[p_0 - \frac{mx}{s} \right] \left[\frac{s-x}{\{y^2 + (s-x)^2\}^2} + \frac{x}{(y^2 + x^2)^2} \right] \\ &\quad - \frac{m}{s\pi} \left[\frac{(s-x)^2 \{(s-x)^2 - y^2\}}{\{y^2 + (s-x)^2\}^2} - \frac{x^2(x^2 - y^2)}{(y^2 + x^2)^2} \right].\end{aligned}$$

Somit erhält man nach der Formel $\tau = \int \frac{\partial \sigma_y}{\partial y} dx$.

$$\begin{aligned}(20) \quad \tau &= \frac{y^2}{\pi} \left[p_0 - \frac{mx}{s} \right] \left[\frac{1}{y^2 + x^2} - \frac{1}{y^2 + (s-x)^2} \right] \\ &\quad + \frac{my}{\pi s} \left[y \left\{ \frac{x}{y^2 + x^2} + \frac{s-x}{y^2 + (s-x)^2} \right\} - \arctg \left(\frac{s-x}{y} \right) - \arctg \left(\frac{x}{y} \right) \right].\end{aligned}$$

Das unbestimmte Integral enthält das Glied $\frac{m}{\pi}$. Da aber für $y=0$ τ den Wert Null erwirbt, muss dieser Wert ausgeschaltet werden.

Schliesslich ergibt sich

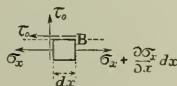
$$\begin{aligned} \frac{\partial \tau}{\partial y} = & \frac{2p_0 y}{\pi} \left[\frac{x^2}{(y^2 + x^2)^2} - \frac{(s-x)^2}{\{y^2 + (s-x)^2\}} \right] \\ & - \frac{2my}{\pi s} \left[x \left\{ \frac{x^2}{(y^2 + x^2)^2} - \frac{(s-x)^2}{\{y^2 + (s-x)^2\}} \right\} \right] \\ & + \frac{m}{\pi s} \left[y \left\{ \frac{x(3x^2 + y^2)}{(y^2 + x^2)^2} + \frac{(s-x)\{3(s-x)^2 + y^2\}}{\{y^2 + (s-x)^2\}^2} \right\} - \operatorname{arctg}\left(\frac{s-x}{y}\right) - \operatorname{arctg}\left(\frac{x}{y}\right) \right]. \end{aligned}$$

und daher infolge der Beziehung $\sigma_x = \int \frac{\partial \tau}{\partial y} dx$

$$\begin{aligned} (21) \quad \sigma_x = & \frac{1}{\pi} \left[p_0 - \frac{mx}{s} \right] \left[\operatorname{arctg}\left(\frac{s-x}{y}\right) + \operatorname{arctg}\left(\frac{x}{y}\right) \right] \\ & - \frac{p_0 y}{\pi} \left[\frac{x}{y^2 + x^2} + \frac{s-x}{y^2 + (s-x)^2} \right] - \frac{my}{\pi s} \left[\ln \left\{ \frac{y^2 + (s-x)^2}{y^2 + x^2} \right\} \right. \\ & \left. - \frac{s(s-x)}{y^2 + (s-x)^2} \right]. \end{aligned}$$

Aufgabe 3. Der Körper sei über die Strecke s der Oberfläche gleichmässig mit Tangentialkräften τ_0 beansprucht.

Fig. 11.



Dass eine Kraft, die wir an der Oberfläche des Körpers wirkend annehmen, sich gleichmässig über die Strecke s verteilen lässt, folglich dass sich Tangentialkraft auf der Flächeneinheit sprungweise am Punkt B ändert, kann praktisch nicht vorkommen. In der Tat stelle man sich am Punkt B ein unendlich kleines Rechteck vor (Fig. 11). Wenn das Rechteck unter der vorgelegten Annahme noch das Gleichgewicht hielte, müsste man die Beziehung

$$\tau_o dx dy = \frac{\partial \sigma_x}{\partial x} dx \frac{(dy)^2}{2},$$

oder

$$\tau_o = \frac{\partial \sigma_x}{\partial x} \frac{dy}{2}$$

erhalten. Dies würde nur dann möglich sein, wenn das Änderungsverhältnis $\frac{\partial \sigma_x}{\partial x}$ dort unendlich gross wäre, was einen Einbruch des Körpers am Punkte B zur Folge haben müsste. Davon werden wir uns in Folgendem rein mathematisch überzeugen können.

Wir nehmen für die Funktion τ den allgemeinen Ausdruck

$$\tau = \int_0^x [A_1 + A_2 y] e^{-xy} \cos ax \cdot \psi(a) da$$

an. Wir haben die Bedingung

$$\int_0^{\infty} \tau dy = 0.$$

Danach ergibt sich

$$\int_0^{\infty} \int_0^{\infty} [A_1 + A_2 y] e^{-xy} \cos ax \cdot \psi(a) da dy = 0$$

oder

$$\int_0^{\infty} \frac{1}{a} \left[A_1 + \frac{A_2}{a} \right] \cos ax \cdot \psi(a) da = 0.$$

Dies fordert

$$A_2 = -a A_1.$$

Somit lautet der Ausdruck für τ

$$\tau = \int_0^s [1 - ay] e^{-xy} \cos ax \cdot \psi(a) da.$$

Bezeichnet man wie in den vorigen Fällen mit $f(x)$ die der Kraftverteilung in der Oberfläche des Körpers entsprechende Funktion, so erhält man ganz wie in der Aufgabe 1

$$\tau = \frac{2}{\pi} \int_0^\infty \int_0^\infty [1 - ay] e^{-xy} f(t) \cdot \cos ax \cos at \, dt \, dx.$$

Die Integration nach t lässt sich ausführen:

$$\int_0^\infty f(t) \cdot \cos at \, dt = \int_0^{\frac{\pi}{2}} \tau_o \cos at \, dt = \frac{\tau_o \sin \frac{as}{2}}{a}.$$

Und man findet

$$(22) \quad \tau = \frac{\tau_o}{\pi} \left[\arctg \left(\frac{\frac{s}{2} + x}{y} \right) + \arctg \left(\frac{\frac{s}{2} - x}{y} \right) - y \left\{ \frac{\frac{s}{2} + x}{y^2 + \left(\frac{s}{2} + x \right)^2} + \frac{\frac{s}{2} - x}{y^2 + \left(\frac{s}{2} - x \right)^2} \right\} \right].$$

Ferner folgt:

$$\frac{\partial \tau}{\partial y} = -\frac{2\tau_o}{\pi} \left[\frac{\left(\frac{s}{2} + x \right)^3}{\left\{ y^2 + \left(\frac{s}{2} + x \right)^2 \right\}^2} + \frac{\left(\frac{s}{2} - x \right)^3}{\left\{ y^2 + \left(\frac{s}{2} - x \right)^2 \right\}^2} \right].$$

Und infolge der Beziehung $\sigma_x = \int \frac{\partial \tau}{\partial y} dx$

erhält man unter Weglassung einer additiven Konstanten

$$(23) \quad \sigma_x = \frac{\tau_o}{\pi} \left[y^2 \left\{ \frac{1}{y^2 + \left(\frac{s}{2} - x \right)^2} - \frac{1}{y^2 + \left(\frac{s}{2} + x \right)^2} \right\} - \ln \left\{ \frac{y^2 + \left(\frac{s}{2} + x \right)^2}{y^2 + \left(\frac{s}{2} - x \right)^2} \right\} \right].$$

Setzt man darin $y=0$, so findet man für die Oberfläche des Körpers den Ausdruck

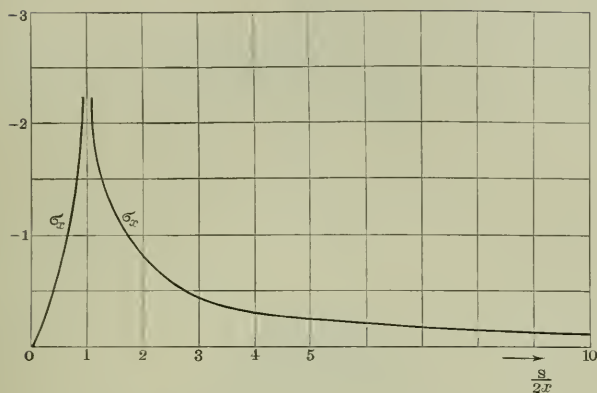
$$(24) \quad \sigma_x = \frac{\tau_o}{\pi} \ln \left[\frac{\frac{s}{2x} - 1}{\frac{s}{2x} + 1} \right]^2.$$

Es geht hervor, dass auf der Oberfläche des Körpers für $x=0$, σ_x verschwindet, während es für $x=\frac{s}{2}$ unendlich gross wird. Jenseits dieser Grenze nimmt σ_x mit x , von unendlich gross an, seinem absoluten Wert nach ab.

Bei numerischer Berechnung wird die folgende Tabelle gute Dienste leisten [vgl. Fig. 12]:

$\frac{s}{2x}$	$\left[\frac{s}{2x}-1\right]^2$	$\left[\frac{s}{2x}+1\right]^2$	$\ln\left[\frac{s}{2x}-1\right]^2$	$\ln\left[\frac{s}{2x}+1\right]^2$	$\frac{1}{\pi} \ln \left[\frac{\frac{s}{2x}-1}{\frac{s}{2x}+1} \right]^2$
0	1	1	0	0	0
0,1	0,81	1,21	-0,21072 10	0,19062 04	-0,12775 1
0,2	0,64	1,44	-0,44628 71	0,38464 31	-0,25812 7
0,3	0,49	1,69	-0,71334 99	0,52472 85	-0,39409 3
0,4	0,36	1,96	-1,02165 31	0,67294 45	-0,53940 7
0,5	0,25	2,25	-1,38629 44	0,81093 02	-0,69939 8
0,6	0,16	2,56	-1,83258 15	0,94000 73	-0,88254 2
0,7	0,09	2,89	-2,40794 56	1,06125 65	-1,0428 1
0,8	0,04	3,24	-3,21887 53	1,17557 33	-1,39879 7
0,9	0,01	3,61	-4,60517 02	1,28370 78	-1,87448 8
1,0	0	4,00	$-\infty$	1,20412 00	$-\infty$
1,2	0,04	4,84	-3,21887 58	1,57691 47	-1,52654 8
1,4	0,16	5,76	-1,83258 15	1,75003 75	-1,14067 7
1,6	0,36	6,76	-1,02165 13	1,91102 29	-0,93349 9
1,8	0,64	7,84	-0,44628 71	2,05923 88	-0,79753 4
2,0	1,00	9,00	0	2,19722 46	-0,79939 8
2,5	2,25	12,25	0,81093 02	2,50552 59	-0,53940 7
3,0	4,00	16,00	1,38629 44	2,77258 87	-0,44125 5
3,5	6,25	20,25	1,83258 15	3,00815 46	-0,37419 7
4,0	9,00	25,00	2,19722 46	3,21887 58	-0,32520 2
4,5	12,25	30,25	2,50552 59	3,40949 62	-0,28774 3
5,0	16,00	36,00	2,77258 87	3,58351 89	-0,25812 7
6,0	25,00	49,00	3,21887 58	3,89182 03	-0,21420 5
7,0	36,00	64,00	3,58351 89	4,15788 31	-0,18314 4
8,0	49,00	81,00	3,89182 02	4,39444 92	-0,15999 2
9,0	64,00	100,00	4,15888 31	4,60517 02	-0,14205 8
10,0	81,00	121,00	4,39444 92	4,79579 05	-0,12775 1

Fig. 12.



Schliesslich, da

$$\frac{\partial \tau}{\partial x} = \frac{2\tau_0 y}{\pi} \left[\frac{\left(\frac{s}{2} + x\right)^2}{\left\{y^2 + \left(\frac{s}{2} + x\right)^2\right\}^2} - \frac{\left(\frac{s}{2} - x\right)^2}{\left\{y^2 + \left(\frac{s}{2} - x\right)^2\right\}^2} \right]$$

ist, bekommt man

$$(25) \quad \sigma_y = \int \frac{\partial \tau}{\partial x} dy = \frac{\tau_0}{\pi} \left[\frac{\left(\frac{s}{2} - x\right)^2}{y^2 + \left(\frac{s}{2} - x\right)^2} - \frac{\left(\frac{s}{2} + x\right)^2}{y^2 + \left(\frac{s}{2} + x\right)^2} \right].$$

Aus den Formeln erhellt, dass die Funktionen σ_x und σ_y , die für positive x stets negativ sind, mit x ihre Vorzeichen wechseln, indem sie ihren absoluten Werte nach unverändert bleiben. Daraus geht hervor, dass, weil τ immer positiv bleibt, die Hauptspannungslinien in den beiden Feldern auf der y -Achse gänzlich symmetrisch bezüglich derselben verlaufen. Die Hauptspannungen

selbst aber besitzen in zwei entsprechenden Punkten beider Felder immer entgegengesetzte Vorzeichen.

Fig. 13, 14, 15 stellen die Funktionswerte σ_y , σ_x , und τ für $s=600$ cm. Fig. 16 bezeichnet den Verlauf der Hauptspannungslinien für denselben Fall.

Fig. 13.

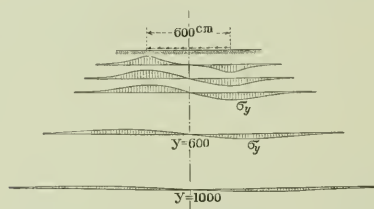


Fig. 14.

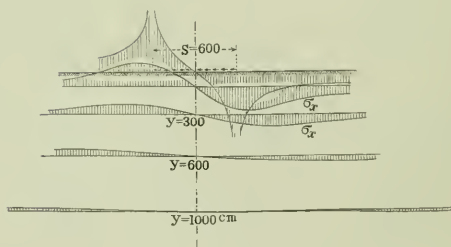


Fig. 15.

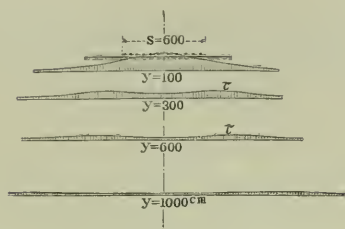
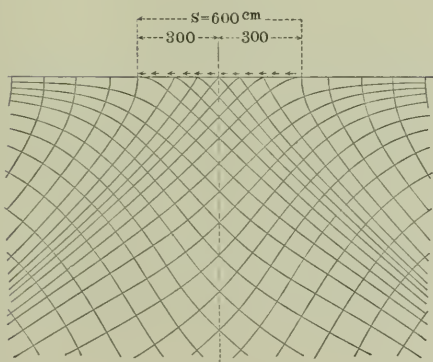


Fig. 16.



Fukuoka, im Juni 1921.

X-Ray Examination of Inner Structure of Strained Metals. I. Chiefly on Copper Wires.

By

Prof. Akimasa Ono, *Kogakuhakushi.*

(Received September 14, 1921)

Introductory.

The X-ray interference phenomena as first analytically investigated by Laue¹⁾ and experimentally brought to light by Friedrich and Knipping, have been extensively studied by various methods and in regard to various points by many authors, among whom W. H. and W. L. Bragg²⁾ may be mentioned. The original investigations performed by Laue and Bragg, as is well known, were concerned with the single crystal, i.e. a system of atoms regularly arranged in a similar manner at any point of the specimen.

Later on, it was shown by Debye and Scherrer³⁾, and also by Hull⁴⁾, that any irregular group of small crystals may be taken as the specimen for investigating the structure of the individual

1) Ann. d. Physik, 41, 1913, p. 971.

2) X-Rays and Crystal Structure.

3) Phys. Zeits., 17, 1916, p. 277.

4) Phys. Rev., 10, 1917, p. 661.

crystals constituting it, or for the analysis of the incident ray.

The latter method was recently used for investigating the structure of various metals, such as chromium, molybdenum, cobalt, nickel, etc. by Hull¹⁾, and specially iron and steel by Westgren.²⁾ Thus the specimens used in the experiments above mentioned were either single crystals of appreciable size or irregular groups of minute ones.

Besides, we have many substances of some regularity, e.g. fibrous and lamellar specimens, which have not escaped the notice of investigators. So fibrous substances in nature were made the subjects of experiments by Nishikawa and his co-worker³⁾, while rolled sheets of metals were experimented on by Keene⁴⁾, Broglie⁵⁾, Nishikawa and his co-workers^{3) 6)}.

In the present note the interference figures obtained from several copper wires will be reported with special regard to the crystal arrangement in the strained copper. The experiment shows that a copper rod constituted of an irregular mass of small crystals, on being drawn into wire takes a fibrous structure, the arrangement of crystal lattices being transformed into a state of axial regularity. The result may probably elucidate the change of properties such as hardness and strength to some degree. Further the interference figures obtained from other metals will be briefly considered.

1) Phys. Rev., 17, 1921, p. 571.

2) Iron and Steel Inst., May Meeting, 1921. Reprinted in Engineering, June, 1921, p. 727.

3) Proc. Tokyo Math. Phys. Soc., 7, 1913-1914, p. 131.

4) Phil. Mag., 1913, II, p. 712.

5) Comptes Rendus, 1914, p. 333.

6) Phys. Rev., 15, 1920, p. 33.

Experiment.

The specimens used in the present experiment were chiefly copper wires, of which the particulars will be given below. Besides, low carbon steel specimens taken from the laboratory store of fractured pieces, and iron, copper, and alloy tubes of unknown origin were also examined.

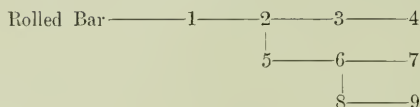
The copper wires used were made of electrolytic copper, and they had been stored in the laboratory over four years. There were two series of specimens, *A* and *B*, of which the former consisted of nine pieces, and the latter of ten pieces. Of the series *A*, the piece No. 1, 7.55 mm in diameter was the product after one drawing from rolled bar, and the pieces Nos. 2, 3, and 4 were of successively smaller diameters corresponding to the various stages of drawing, the final diameter being about 2.38 mm. The piece No. 5 was an annealed specimen of No. 2, and Nos. 6 and 7 were obtained by further drawing of No. 5 to nearly the same dimensions as Nos. 3 and 4, respectively. Lastly, the piece No. 8 was obtained by annealing No. 6, and No. 9 by drawing the former to the same dimension as the final state of the foregoing specimens.

Again, as to the series *B*, No. 1 was the rolled bar of 12.4 mm diameter, and the successive pieces down to No. 5 were taken from various stages of drawing. Further, No. 6, 7, and 8, originated from No. 3, while Nos. 9 and 10 came from No. 7, by annealing and drawing. All the final products of the series *B* were of practically the same diameter, not only among themselves, but also as compared with the series *A*.

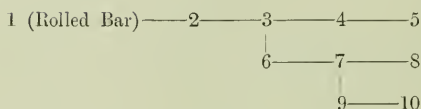
The above explanation may be simply expressed in the following schemes, in which the horizontal line represents the

drawing, and the vertical line refers to the annealing.

A—Series.



B—Series.



The steel specimens already mentioned contained 0.037% C, and they were formerly experimented on for investigating the effect of straining on the impact resistance of metals¹⁾. The treatment suffered by the specimens may be understood from the remarks following.

- N* 1 taken from the locally contracted part of a test piece previously annealed at 900°C. for half an hour before turning, and fractured by pull, the amount of strain being about 50%.
- N* 2 taken from Charpy impact test piece previously annealed in the same manner as *N* 1.
- N* 9 taken from Charpy impact test piece prepared from the material annealed and strained, the amount of strain being 9%.
- N* 10 taken from Charpy impact test piece prepared from the material annealed, strained, and again annealed, the amount of strain being same as *N* 9.

1) This memoir, Vol. II, No. 1, 1920.

These steel specimens had been kept for about five years since the mechanical test was made.

Now the specimen for the present experiment was prepared in the form of a thin plate by cutting the material along its longitudinal section; thus the direction of the pull or drawing coincided with the longitudinal axis of the specimen. The thickness of the slice was either 0.20 or 0.25 mm; most of the specimens taken from the copper wires and the low carbon steel were finished to the thickness of 0.20 mm (Figs. 1-15), but some of the copper specimens were finished to 0.25 mm (Fig. 16-18).

A Coolidge tube was operated with a transformer, and the white ray issuing from the tungsten target was passed through holes in lead diaphragms and allowed to penetrate the specimen in the direction normal to its plane. The diameter of the holes was generally 2 mm, and the distance between the specimen and the photographic plate was either 47 or 85 mm, excepting the cases of Figs. 16-18, in which the hole diameter was 4 mm, and the plate distance was 88 mm. The voltage measured by the spark gap of needle points was about 135 mm corresponding to some 94 000 volts, and the current was 3 to 4 milliamperes. The time of exposure varied from $1\frac{1}{2}$ to 9 hours according to condition.

Results of Experiment.

The typical interference figures obtained in the experiment are reproduced in the plates, Figs. 1 to 18, and we observe the following points.

(1) The interference figure obtained from metal pulled or drawn as above stated, shows six bands radiating from the cen-

tral spot, though there are sometimes other less definite images mingled with the above, and the said pattern appears not to be the final state of the change at least in the case of copper wires, as noted in the next paragraph.

(2) As the drawing proceeds by degrees, the separation of the radiating bands or lines becomes marked, and the figure becomes clearer, as can be seen in the series of photographs taken from the copper wires, series *A*. The more detailed particulars to be observed from the figures will be given in the paragraph (4).

(3) The figure obtained from the material annealed after straining, shows numerous spots indicating the growth of individual crystals by aggregation. In the case of the low carbon steel with coarse grain due to the previous treatment, the spots are very vivid, and the distribution is quite irregular around the centre; while a certain regularity still remains in the copper specimen annealed at 1000°C . for 30 min.

(4) The angles between the radiating lines and the central band were measured with the result as shown in the following table, in which 1, 2, 3, 4 represent the radiating lines beginning from the ones next to the central line, Fig. 19.

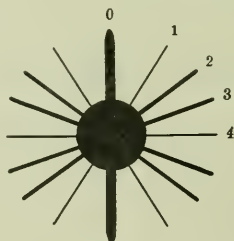


Fig. 19.

Radiating Lines	1	2	3	4
A—Series	33-35	54-55	70-71 $\frac{1}{2}$	90 degrees
B—Series	33-33 $\frac{1}{2}$	54-55	70-71	90 „

The higher figures in the table refer to the more highly drawn specimens. Further, the angle observed in the specimen successively drawn without intermediate annealing, appears to be greater than the angle determined from the specimen drawn to that state with the intervention of heating. These remarks are added here as they were found probable though the determination of the angles was accompanied by an error comparable with the difference which might really exist.

Interpretation of the Result.

The above experimental result shows that the individual crystals in the strained material, are partly or wholly so arranged, that the normal to the reflecting plane is inclined at definite angles to the direction of pull or drawing, the distribution being quite irregular in other respects. This is verified by the interference figures obtained in other different positions of the specimens than those reproduced in the present report. The figure is essentially similar when the plane of the specimen is rotated about its longitudinal axis, which remains perpendicular to the direction of the incident ray, but the figure changes when the rotation takes place about the transverse axis, the longitudinal axis being then inclined at a certain angle.

The analysis of the interference figure which might be expected in such crystal systems as the present ones, was already

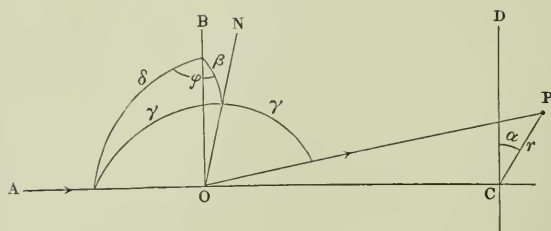


Fig. 20.

given by Nishikawa and his collaborator, but a brief account of the analytical determination is given below, as the formulæ will be found necessary to the development of the following treatment.

Referring to Fig. 20, AO is the direction of the incident ray making an angle δ with the axis of the specimen OB , and OP is the ray reflecting at a plane with the normal ON , the angles between the rays and the normal being γ . The position of the normal and therefore also of the plane is determined by the angles β and φ . The plane DCP representing the photographic plate stands normally to OC , and the coordinates of the point P in the plane are represented by a and r . Now

$$\cos \gamma = \cos \beta \cos \delta + \sin \beta \sin \delta \cos \varphi. \quad (1)$$

If we put $r = \frac{\pi}{2} - \theta$, then θ is the so-called glancing angle, and it is related to the wave length and the lattice constant by the well known equation $n\lambda = 2d \sin \theta$. Thus there is a certain angle φ for the definite angles β , γ , and δ , the reflection taking place at the plane thus determined.

To find α , we have

$$\cos \beta = \cos \gamma \cos \delta + \sin \gamma \sin \delta \cos \alpha, \quad (2)$$

and r is given by the following equation.

$$r = -L \tan 2\gamma. \quad (3)$$

Particularly, when $\delta = \frac{\pi}{2}$ as in the present case,

$$\cos \gamma = \sin \beta \cos \varphi, \quad (1a)$$

and

$$\cos \beta = \sin \gamma \cos \alpha. \quad (2a)$$

The locus of P may be traced out by giving various values to φ or r , and finding α and r . From the above expressions it is manifest that if $\gamma = \delta = \frac{\pi}{2}$, then $r = 0$, $\alpha = \beta$, and $\frac{da}{dr} = 0$. Further we see that $\frac{da}{dr} < 0$ for $\gamma < \frac{\pi}{2}$. So the locus is approximately in the direction of $\alpha = \beta$ in the vicinity of C , and it gradually curves towards the axis CD .

From the above statement it follows that the angles determined from the photographs by measuring the inclinations of the radiating lines of certain length give approximately the values of β . Here it is to be mentioned that the approximation is quite good in the radiating lines 2 and 3, which are nearly straight, while the measured angle will be rather smaller than the true value of β in the case of the line 1, as the curvature becomes sensible. Thus the inclinations of reflecting planes being determined with certain degrees of correctness, it is not without interest to compare the observed angles with the calculated values by assuming the probable arrangement of crystal lattices in the specimen.

The crystal structure of natural copper specimens was spec-

troscopically investigated by W. L. Bragg¹⁾ with the result that the copper atoms were arranged on a face-centred cubic lattice. So let us consider if such a system of planes can be constructed as to conform to the observed values of β .

Taking the origin at the centre of a corner atom, and the rectangular coordinate axes along the edges of the cubic lattice, the direction cosines of the normal to a plane with the indices h_1, h_2, h_3 , are

$$l = \frac{h_1}{\sqrt{h_1^2 + h_2^2 + h_3^2}}, \quad m = \frac{h_2}{\sqrt{h_1^2 + h_2^2 + h_3^2}}, \quad n = \frac{h_3}{\sqrt{h_1^2 + h_2^2 + h_3^2}}, \quad (4)$$

respectively.

Thus the angle between the normal (l, m, n) and a certain axis with the direction cosines l', m', n' being denoted by β , we have

$$\cos \beta = \frac{h_1 l' + h_2 m' + h_3 n'}{\sqrt{h_1^2 + h_2^2 + h_3^2}}. \quad (5)$$

In order to know whether the angles β calculated by this expression may coincide with those already found (p. 247), let us provisionally take simple cases such as

- (i) $l' = m' = 0, \quad n' = 1,$
- (ii) $l' = 0, \quad m' = n' = \frac{1}{\sqrt{2}},$
- (iii) $l' = m' = n' = \frac{1}{\sqrt{3}}.$

In these cases the fibre direction of the specimen coincides with (i) an edge of the cubic lattice, (ii) a diagonal of the

1) Phil. Mag., 1914, II, p. 355.

face, and (iii) a diagonal of the cube, and the expression (5) becomes

$$\left. \begin{aligned} \text{(i)} \quad \cos \beta &= \frac{h_3}{\sqrt{h_1^2 + h_2^2 + h_3^2}}, \\ \text{(ii)} \quad \cos \beta &= \frac{h_2 + h_3}{\sqrt{2} \sqrt{h_1^2 + h_2^2 + h_3^2}}, \\ \text{(iii)} \quad \cos \beta &= \frac{h_1 + h_2 + h_3}{\sqrt{3} \sqrt{h_1^2 + h_2^2 + h_3^2}}, \end{aligned} \right\} \quad (6)$$

respectively.

Assigning certain simple values to h_1, h_2, h_3 , we obtain the values of β as shown in the following table.

h_1, h_2, h_3	(i) $\cos \beta$ β	(ii) $\cos \beta$ β	(iii) $\cos \beta$ β
1, 0, 0	0, 1 $90^\circ, 0$	0, $\frac{1}{\sqrt{2}}$ $90^\circ, 45^\circ$	$\frac{1}{\sqrt{3}}$ $54^\circ 44' 10''$
1, 1, 0	0, $\frac{1}{\sqrt{2}}$ $90^\circ, 45^\circ$	$\frac{1}{2}, 1$ $60^\circ, 0$	$\sqrt{\frac{2}{3}}$ $35^\circ 15' 50''$
1, $\bar{1}$, 0	0, $\frac{1}{\sqrt{2}}$ „	$\frac{1}{2}, 0$ $60^\circ, 90^\circ$	0 90°
1, 1, 1	$\frac{1}{\sqrt{3}}$ $54^\circ 44' 10''$	$\sqrt{\frac{2}{3}}$ $35^\circ 15' 50''$	1 0
1, 1, $\bar{1}$	$\frac{1}{\sqrt{3}}$ „	0, $\sqrt{\frac{2}{3}}$ $90^\circ, \text{ „}$	$\frac{1}{3}$ $70^\circ 31' 40''$

Comparing these three cases we observe that the first two cases fail to give the radiating lines in the observed directions so far as the above planes are concerned. Of course it is to be noted that there are other planes with the normals in the proper directions. For example, consider the planes (1, 1, 2) and (2, 2, 1); then, $\frac{h_3}{\sqrt{h_1^2 + h_2^2 + h_3^2}}$ becomes equal to $\sqrt{\frac{2}{3}}$ and $\frac{1}{3}$, i.e. $\beta = 35^\circ 15' 50''$ and $70^\circ 31' 40''$, respectively. But owing to the

rare density of atoms in such planes, the reflection must be naturally weak in comparison with those already considered. Moreover, the radiating lines corresponding to $\beta=45^\circ$ can not be observed in the photographs.

Thus it is probable that the trigonal axis coincides with the longitudinal direction or the fibre direction of the specimen. Whether this is really the case may be tested by the spectrographic investigation, which is under preparation, but we must be satisfied for the present with the following verification by calculation of the intensity.

Intensity of Reflection.

In order to verify the preceding inference, let us now calculate the relative intensity of reflection, and compare the calculated value with the intensity of the bands observed in the photographs. Debye¹⁾ expressed the intensity of reflection in his experiment of powdered crystals as the product of the factors relating to (1) scattering, (2) interference, (3) temperature effect, (4) irregularity of incident ray. Lastly, the product should be multiplied by the number of the co-working planes.

In the present calculation the approximate comparison of the intensities of different bands is aimed at, and therefore the factors (1) and (3) are disregarded. Next, the interference factor (2) gives obviously no influence on the intensity of the reflection of the planes shown in the table on p. 257, in which the indices 2, 2, 0; 2, 0, 0; 2, $\bar{2}$, 0 are written in the places of 1, 1, 0; 1, 0, 0; 1, $\bar{1}$, 0, respectively, in order to refer to the primary interference of adjacent planes.

1) Phys. Zeits., 18, 1917, p. 487.

Now the factor (4), or Lorentz-factor as it is called, originates, as already briefly remarked, from the assumption that the incident ray is neither absolutely monochromatic nor truly parallel, and it is of the form $\frac{1}{h_1^2 + h_2^2 + h_3^2}$. As $n\lambda = \frac{2a \sin \theta}{\sqrt{h_1^2 + h_2^2 + h_3^2}}$, a being the side of the elementary cube, this factor may be written in the form $\frac{1}{\sin^2 \theta}$ if desired. In the present case the incident ray is considered to be parallel though not strictly correct, and special calculation is made to account for the variation of the intensity with the type of planes.

The number of co-working planes giving the reflection in the same direction will be first considered. By the last of the equations (6) we learn that there is generally a set of values h_1, h_2, h_3 for a definite value of β , and the number of planes is

- (a) 6, if all of the indices are different,
- (b) 3, if two of the indices are equal,
- (c) 1, if all of the indices are equal.

But two parallel planes should be counted once in determining the number under consideration. For example, the planes $2, \bar{2}, 0$ and $\bar{2}, 2, 0$ are indifferent. The number μ thus determined is given in the fourth column of the table already mentioned.

Next referring to the interference factor, we see that the intensity of the radiation from a number of points x, y, z belonging to the cubic system depends on the square of the absolute value of

$$\sum e^{\frac{2\pi i}{a} [xh_1 + yh_2 + zh_3]}.$$

For any point on the reflecting plane (h_1, h_2, h_3) , we have the following equation.

$$xh_1 + yh_2 + zh_3 = a.$$

Accordingly, the above expression is equal to the number of atoms lying on a definite plane multiplied by the number of planes to be considered, and it gives rise to no influence on the relative intensity, if the summation extends to all the planes. Here we take, as assumed by some authors¹⁾, a definite number of planes to be effective in the interference. Then disregarding the constant factor, the intensity may be considered to depend on the square of the number of atoms lying on the plane in each cube.

The number of atoms N just mentioned is equal to the product of the plane density of atoms and the area of the plane. Thus

$$N = AMd = \frac{AMa}{\sqrt{h_1^2 + h_2^2 + h_3^2}}, \quad (7)$$

where A = the area of the plane in each cube,

M = the number of atoms in unit volume.

The value of M being equal to $4/a^3$ in the face-centred cubic lattice, the values of N will be found as shown in the fifth column of the table.

Now the important thing to be considered is the effect of the variation of β and φ . The latter takes various values as the nature of the structure indicates, while the former may also be inconstant owing to the irregularity of the fibrous structure.

In the following calculation the infinitesimals $d\beta$ and $d\varphi$ are used instead of $\Delta\beta$ and $\Delta\varphi$, though the variation is not continuous, and we take $d\beta d\varphi$ as the measure of the number of

1) Webster, Compton, and also Friedel. Cp. Comptes Rendus, 1919, p. 1147.

reflecting planes, of which the normals are in the region as signified by $d\beta$ and $d\varphi$.

If we now consider the influence of the small changes of β and φ on the variation of r and a , then

$$dr = \frac{\partial r}{\partial \beta} d\beta + \frac{\partial r}{\partial \varphi} d\varphi,$$

$$da = \frac{\partial a}{\partial \beta} d\beta + \frac{\partial a}{\partial \varphi} d\varphi.$$

In the latter equation γ is considered as invariable, as r is provisionally kept constant. Accordingly, if we remember that $\left(\frac{\partial a}{\partial \varphi}\right)_\gamma = 0$, we have $da = \left(\frac{\partial a}{\partial \beta}\right)_\gamma d\beta$, $\left(\frac{\partial a}{\partial \beta}\right)_\gamma$ and $\left(\frac{\partial a}{\partial \varphi}\right)_\gamma$ being the differential coefficients in which γ is considered as constant. Thus it follows that the elementary area $r dr da$ corresponds to the product $r \frac{\partial r}{\partial \varphi} \left(\frac{\partial a}{\partial \beta}\right)_\gamma d\beta d\varphi$.

From (2a)

$$\left(\frac{\partial a}{\partial \beta}\right)_\gamma = \frac{\sin \beta}{\sin \alpha \sin \gamma},$$

and from (3) and (1a)

$$\begin{aligned} \frac{\partial r}{\partial \varphi} &= \frac{\partial r}{\partial \gamma} \frac{\partial \gamma}{\partial \varphi} = -2L \frac{\sin \beta \sin \varphi}{\cos^2 2\gamma \sin \gamma} \\ &= -2L \frac{\sqrt{\sin^2 \gamma - \cos^2 \beta}}{\cos^2 2\gamma \sin \gamma}. \end{aligned}$$

By introducing (2a),

$$\frac{\partial r}{\partial \varphi} = -2L \frac{\sin \alpha}{\cos^2 2\gamma}.$$

Thus

$$\frac{\partial r}{\partial \varphi} \left(\frac{\partial a}{\partial \beta}\right)_\gamma = -2L \frac{\sin \beta}{\sin \gamma \cos^2 2\gamma}.$$

As the intensity may be considered to be inversely proportional to the absolute value of the ratio $r \, dr \, da / d\beta \, d\varphi = r \frac{\partial r}{\partial \varphi} \left(\frac{\partial a}{\partial \beta} \right)_r$, we see that it varies as $\frac{\sin \gamma \cos^2 2\gamma}{2L r \sin \beta}$ for a definite value of λ .

So far as the foregoing consideration is concerned, the intensity appears to depend among other things on the expression

$$\begin{aligned} I &= \frac{\mu N^2}{2L^2} \frac{\sin \gamma \cos^2 2\gamma}{\sin \beta} = -\frac{\mu N^2}{4L^2} \frac{\cos^3 2\gamma}{\sin \beta \cos \gamma} \\ &= \frac{\mu N^2}{4L^2} \frac{\cos^3 2\theta}{\sin \beta \sin \theta}. \end{aligned} \quad (8)$$

Here we may put $\cos^3 2\theta = \sim 1$ for small values of θ . The error introduced by this simplification is inconsiderable. For example, with $2\theta = 10^\circ$, $\cos^3 2\theta = 0.955$, and therefore the error introduced in the relative value of the intensity will be much less than 5%. Thus we have

$$\begin{aligned} I &= \frac{\mu N^2}{2L^2 \sin \beta} \frac{a}{n\lambda \sqrt{h_1^2 + h_2^2 + h_3^2}} \\ &= c \frac{\mu N^2}{\sin \beta \sqrt{h_1^2 + h_2^2 + h_3^2}}, \end{aligned} \quad (9)$$

c standing for the quantity $\frac{a}{2L^2 n\lambda}$, which is constant in the present case.

Lastly, it is to be added that the above measure of the intensity holds good for any bands so long as $\beta < \frac{\pi}{2}$, but when $\beta = \frac{\pi}{2}$, the intensity will be doubled as the four branches of the curve coincide in the two lines radiating in opposite directions.

Moreover, in the particular case when $\beta = 0$, the intensity becomes infinity, the image being contracted to a single point,

which can not really exist. Thus we take the point $r=0.224 L$, $\alpha=0$ of the central band. This ratio of $\frac{r}{L}$ corresponds to $r=19$ mm in the photographs of A-Series and 10.5 mm in B-Series, respectively, and such a point marks the boundary of the comparatively intense region of the band.¹⁾ With the above ratio of $\frac{r}{L}$,

$$2\gamma=167^{\circ}20', \quad \gamma=83^{\circ}40',$$

and as $\alpha=0$,

$$\beta=\frac{\pi}{2}-\gamma=6^{\circ}20'.$$

This value of β is conventionally used in the calculation of the intensity instead of putting $\beta=0$. In any case the intensity of the plane under consideration is most intense.

With the preceding remarks we may introduce the result of the numerical calculation as shown in the following table.

Plane No.	Indices h_1, h_2, h_3	Inclination of Normal β	Number of Planes μ	Number of Atoms N	Intensity μN^2
					$\sin \beta \sqrt{h_1^2 + h_2^2 + h_3^2}$
0	1 1 1	$\begin{smallmatrix} (0) \\ 6^{\circ}20' \end{smallmatrix}$	1	2	$\begin{smallmatrix} (\infty) \\ 20.9 \end{smallmatrix}$
1	2 2 0	$35^{\circ}16'$	3	1	1.8
2	2 0 0	$54^{\circ}44'$	3	2	7.4
3	1 1 $\bar{1}$	$70^{\circ}32'$	3	2	7.4
4	2 $\bar{2}$ 0	90°	3	1	2.1

Strictly speaking, we have to compare the intensities observed at such a point of the different bands as corresponding to a

1) The corresponding value of λ is 0.460 A.U., i.e. rather near to the critical absorption wave length of silver.

definite wave length, i.e. the point for which

$$\cos \gamma = \sin \theta = \text{constant} \sqrt{h_1^2 + h_2^2 + h_3^2}.$$

Still, judging from the general feature of the different bands, we may conclude without this calculation that the values as tabulated in the last column of the above table appear to be in the right order of the relative intensities observed in the photographs. As the foregoing calculation is, however, based on the several assumptions, it is desirable to test the result by the further investigation.¹⁾

Change of Structure.

The ultimate nature of the change taking place in the structure of metals by permanent straining is still a riddle not clearly solved, and there are several hypotheses for explaining the change. The amorphous hypothesis postulates the breaking up of the crystal lattice situated at the slip planes of grains, and the formation of an amorphous substance by the atoms irregularly distributed there. By attributing to this layer of material the properties conceived of as characteristic of an amorphous body, the change in the nature of the material is explained.

On the other hand, the slip plane hypothesis, as it may be called, considers that the formation of the slip plane itself is the cause of the change in the properties, and this effect is ascribed to the fineness of the grains without reference to any change in their final structure.

The present experiment is chiefly concerned with the

1) The test with monochromatic ray performed after the present experiment was made, seems also to confirm the inference made in the preceding section.

geometrical consideration of strained materials, and it does not aim at the systematic investigation of the subject now touching, but the fact that the orientation of the crystal lattice changes into a more regular condition by straining may likely throw light on the alteration of the properties as strength, hardness, etc.

As to the amorphous hypothesis, it is to be added that notwithstanding the question whether amorphous material may or may not effect the interference figure, the present photographs can not claim to decide positively the existence or non-existence of the said substance in the strained material, but considering the circumstance that the fibrous structure predominates as the straining proceeds, it seems to be natural to seek the cause of the change in the altered structure of the crystal aggregate instead of considering the hypothetical amorphous body as the site of the change.

Summary.

1. X-ray examination was performed on strained metals, chiefly on copper wires, and also on those annealed after straining, by taking interference figures, which show that the crystal lattices form fibrous structure in the pulled or drawn state, and this structural rearrangement proceeds as the degree of strain increases.

2. By finding the inclinations of several simple planes in the face-centred cubic lattice, and calculating the relative intensity of rays reflected by the planes, it was shown that a group of the said kind of lattice with the trigonal axis in the longitudinal direction of the specimen, gives rise to the pattern obtained in the experiment.

3. A brief reference was added to the existing hypotheses for the structural change of the strained material, and it was suggested that changes in the lattice arrangement of crystals afford a natural explanation of the change of properties observed in metals that have been subjected to strain.

Fukuoka, September, 1921.

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Fig. 1.

A 1.



Fig. 2.

A 2.



Fig. 3.

A 3.



Fig. 4.

A 4.

Figs. 1-4. Copper wires at various stages of drawing.

260^b



Fig. 5.

A 5.



Fig. 6.

A 6.



Fig. 7.

A 7.

Fig. 5. Copper wire annealed.
Figs. 6 and 7. Copper wires annealed and drawn.

260°



Fig. 8.

A 8.



Fig. 9.

A 9.



Fig. 10.

B 1.



Fig. 11.

B 5.

-
- Fig. 8. Copper wire annealed.
Fig. 9. Copper wire annealed and drawn.
Fig. 10. Rolled copper before drawing.
Fig. 11. Copper wire at the end of drawing.

260^d



Fig. 12.

N 1.



Fig. 13.

N 2.



Fig. 14.

N 9.



Fig. 15.

N 10.

Fig. 12. Iron annealed, drawn out by pull.

Fig. 13. Iron annealed.

Fig. 14. Iron annealed and pulled.

Fig. 15. Iron annealed, pulled, and again annealed.

260²



Fig. 16.

A 2 annealed at 900°C.
for 30 min.



Fig. 17.

A 2 annealed at 900°C.
for 2 hours.



Fig. 18.

A 2 annealed at 1000°C.
for 30 min.

X-Ray Examination of Inner Structure of Strained Metals. II. Copper under Various Kinds of Deformation.

By

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(Received May 10, 1922.)

Preliminary Remarks.

Since the examination of strained metals, chiefly copper wires, was made as reported in the previous paper, the experiments were continued along similar lines, and the results of the examination of wires and tubes, and of several copper rods (strained by extension, compression, and torsion, respectively), are collected in the present report.

The specimen was generally prepared in the form of a thin plate, 0.2 mm in thickness, and a Coolidge tube of tungsten anticathode was used by supplying the current from a transformer, the ray issuing from the tube being allowed to penetrate the specimen in the direction normal to its plane. The holes in the lead diaphragms placed between the tube and the specimen were round in most cases, and the diameter was 2 mm, if not otherwise stated in the following. The photographic plate or film was generally placed at the distance of some 6 cm or 10 cm from the specimen.

In certain series of experiments a tungsten filter was put just before the diaphragm facing the tube. The ray thus filtered

will be spoken of as monochromatic though this is not strictly correct.¹⁾

The potential applied was about 100 k.v., and the current was some 3 to 4 milliamp. The duration of exposure was usually 9 hours in the case of the white ray for a plate distance of 10 cm and a current of 4 milliamp., while it was as long as 18 hours or more in the case of the monochromatic ray for a distance of 6 cm and a current of $3\frac{1}{2}$ milliamp. Here it is to be added that notwithstanding the long exposure the diameter of the holes in the diaphragms was chosen as large as above mentioned, because of the moderate value of the current used in the experiment.

The central part of the plate or film where the unreflected rays may pass, was protected with a small lead cover to prevent undesirable exposure to the intense radiation. The photographs thus taken are reproduced in Figs. 1 to 25.

In preparing the specimens for the present investigation, Mr. Fukuda, assistant of the institute, was chiefly engaged.

Remarks on Copper Wire.²⁾

Before entering into the report of the experimental result obtained from the tubes and the other strained specimens, let us first refer to the interference figure obtained from copper wires by using the monochromatic ray. In the well known formula of the reflection of the ray with the wave length λ at the glancing angle θ , viz., $n\lambda = 2d \sin \theta$, we see that the grating constant d is equal to $a/\sqrt{h_1^2 + h_2^2 + h_3^2}$, a being the side of an elementary cube, i.e. the distance between any two consecutive corner atoms, and

1) A.W. Hull, Phys. Review, 10, 1917, p. 667.

2) Cp. the first report. This memoir, Vol. II, No. 5, 1922, p. 249.

h_1, h_2, h_3 , the indices of the reflecting plane, as defined in the previous paper.

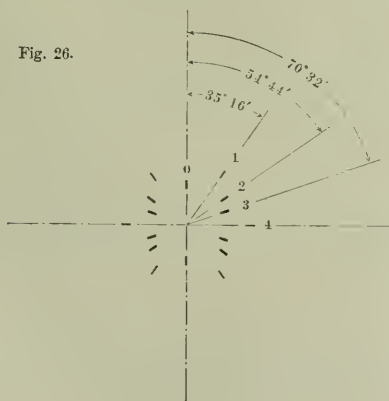
Thus restricting our consideration to the primary reflection, and putting $\gamma = \frac{\pi}{2} - \theta$, we have

$$\cos \gamma = \frac{\lambda}{2a} = \frac{\lambda}{2a} \sqrt{h_1^2 + h_2^2 + h_3^2}.$$

In the present case the value of a is $3.609 \cdot 10^{-8}$ cm, while the value of λ is rather ambiguous, as the ray is not truly monochromatic. If we, however, take $\lambda = 0.2089 \cdot 10^{-8}$ cm corresponding to a_1 line of K-series¹⁾, we can find the radial distance of the respective spot in the interference figure by the equation²⁾

$$r' = -L \tan 2\gamma.$$

Referring to Fig. 26, the spot 0 is a point in the central



band of the white ray diagram, while the symbols 1, 2, 3, and

1) M. Siegbahn, Phil. Mag., 1919, II, p. 639.

2) Cp. the first report.

4 signify the successive spots symmetrically arranged about the vertical and horizontal centre lines. The values of r in the following table correspond to the film distance L of 61 mm.

Interference Points	Indices h_1, h_2, h_3	Grating Angles $\pi/2 - \gamma$	Radial Distances $r = -L \tan 2\gamma$
0	1 1 1	$2^\circ 52' 24''$	0.614 cm
1	2 2 0	$4^\circ 41' 44''$	1.009 „
2	2 0 0	$3^\circ 19' 6''$	0.710 „
3	1 1 $\bar{1}$	$2^\circ 52' 24''$	0.614 „
4	2 $\bar{2}$ 0	$4^\circ 41' 44''$	1.009 „

The interference figures obtained from the copper wires A1 and A4 are reproduced in Figs. 1 and 2. In the former diagram the points above mentioned can not be fully observed, the image being chiefly due to the plane 2, 0, 0, while the latter shows the short bands or spots as might be expected from the white ray diagrams already given in the first report. Referring to Fig. 2 and the like diagrams not reproduced here, the values of r as given in the last column of the above table seem to be in good accord with the experimental result, although the spots of the interference diagram are not naturally sharp enough to allow of accurate measurement, owing to the comparatively large openings of the diaphragms and also to the mixed composition of the incident ray.

Thus there are two kinds of patterns to be observed in copper wires. This has been already pointed out in the previous report, but the inference made there with regard to the rearrangement of the lattice seems to be confirmed by the present experiment. Accordingly, this result may be conveniently referred to in examining the photographs obtained from other kinds of specimens

with which the present report is chiefly concerned.

Lastly, it may be added that the similar experiment with the monochromatic ray was tried with holes of an $\frac{1}{8}$ inch diameter, viz., 1 mm. The bands with higher intensity showed naturally a better result in sharpness, but the image of the fainter bands was too weak to be observed.

Drawn Tubes.

The materials of the tubes experimented on were steel, copper, and brass, and these specimens were made by drawing the material through die holes, the final diameters being as follows.

Material	External Diameter	Internal Diameter
	<i>mm</i>	<i>mm</i>
Steel	3	1.4
Copper	2.6	1.9
Brass	3.2	2.3

A slice of the material was cut from each tube along the longitudinal direction, and the thickness was reduced to about 0.20 to 0.35 mm by filing or polishing. Thus the specimen used in the experiment was of prismatic form, of a curved cross section with or without the plane surface, as shown in Figs. 27 (steel) and 28 (copper and brass).

Fig. 27.



Fig. 28.



All the photographs, Figs. 6, 7, and 8, obtained with white radiations, show one point of similarity namely that there are the six bands radiating from the centre of the figure, the central band coinciding with the direction of drawing, and showing the highest degree of intensity except in Fig. 8. The angles between the bands are nearly equal to 60° . Besides these six bands, there are the intermediate bands far weaker than the former ones in intensity.

The copper specimen was examined with the monochromatic ray, and the photograph obtained, (Fig. 9), shows six vivid spots, of which the middle ones corresponding to the central bands are most intense. Besides, there are less clear figures to be observed in the photograph. In this case the film distance was nearly equal to 60 mm, and the radial distance of the six spots measured from the centre of the figure being constant, it bears with the film distance the same relation as observed in the point No. 2, Fig. 26, due to the plane 2, 0, 0 in the case of the drawn copper wire. Moreover, the six points are arranged at regular intervals, and the general features are similar to Fig. 1, but not to Fig. 2.

The fact that the pattern of the figure is not identical to that of the hard drawn wire (Fig. 2) solicits attention. Whether this is due to the difference in the degree of the drawing, or to the difference in the process of the manufacturing is an open question, but the following series of experiments will throw some light on the behavior of copper under the individual action of extension and compression, and may serve to elucidate the nature of the strain suffered by the drawing, which consists of axial extension and lateral compression.

Extension.

A broken copper test piece was taken from a number of pieces stored in the laboratory. The test piece had been prepared by annealing a rolled bar and reducing the diameter to 20 mm by turning. It had been subjected to the statical pull test last summer. The following data may be here noted for the sake of reference.

Tensile strength referred to the original area of the cross section	= 2200 kg/cm ² ,
Area contraction of the fractured surface	= 75 %,
Breaking extension in 20 cm	= 50 %.

Now two longitudinal sections of the piece were prepared for the present examination, one from the material near the fractured surface and the other from that near one of the gauge marks at a distance of a little more than one half the gauge length from the fractured surface. The extension of the material of the latter section is some 40 %, while that of the former one will be far greater than it.

The material near the fractured section gives under the white radiation the pattern as shown in Fig. 5, essentially similar to the hard drawn wire. This photograph was taken with a hole diameter of 1 mm and at a film distance of 61 mm; thus the width of the band is finer than in other photographs.

Again examining the above specimens with the monochromatic ray, the pattern of the interference figures appears to be again of a similar type to that of the hard drawn wire, the spots arranged in the direction of the extension being much more intense than the others, and the spots corresponding to the points 2 and 3 on p. 263 being clearly observable in the present

case. Thus Fig. 3 shows the pattern obtained from the material near the gauge mark, and Fig. 4 that from the section near the fractured surface, the film distance being equal to 61 mm.

The pattern is different from that of compression as can be seen in the next section, and the type under consideration seems to be the characteristic feature of the extension so far as the examination shows up to the present. In the case of the drawn tubes above reported the action of extension would probably not have been very important compared with the lateral compression. However, the exact knowledge of the process of making being absent, this can not be decided.

Compression.

The compression specimens were taken from a rolled copper rod annealed at 500°C. for 30 min., and they were prepared by squeezing between the compression surfaces of a testing machine in the laboratory. In order to reduce as much as possible the ratio of the final and the initial height of the test piece for a definite dimension of the former, the initial dimensions of the piece should be naturally large, and therefore it was necessary to turn the piece after squeezing it to a certain degree, as the capacity of the testing machine was limited to 30 000 kg. This was the case with specimens Nos. 9 and 10. The dimensions of the piece are shown in the following table.

Specimen No	Initial Diameter <i>mm</i>	Height		Ratio (Final) Height / (Initial) Height
		Initial <i>mm</i>	Final <i>mm</i>	
1	25.50	24.94	14.54	0.583
3	20.00	20.00	8.50	0.425
4	15.00	15.00	4.61	0.307

6	10.00	10.00	1.99	0.199
9	30.01	30.00	2.04	0.068
10	30.00	30.00	2.66	0.089

Of the compression pieces the following sections were examined, Fig. 29.

- (i) Meridian section,
- (ii) Section parallel to a meridian plane,
- (iii) Cross section.

The interference figure obtained from a meridian section by using the white ray shows the six bands symmetrically arranged about the two perpendicular directiones (vertical and horizontal), and the intensity of the central bands is not sensibly different from the other four, Figs. 13 to 21. The photograph shows the structure at the middle part of the said section, if not otherwise stated. The angles between the central and the inclined bands are usually about 55° or a little greater, but it is always less than 60° . There are also the intermediate bands, far weaker than those mentioned above. The type of the pattern remains almost constant for all the specimens with the different degrees of compression.

In the photographs the vertical direction corresponds to the radial direction in which the expansion of the material took place, and the pattern of the figure is similar to that in the

Fig. 29.



case of the drawn tube, except in regard to the angles between the bands.

Here it is to be added that the interference figure obtained at a point near end of the section, i.e. near the external surface of the piece, shows a certain irregularity in the distribution of the radiating bands as can be seen in Figs. 14 and 18. This may be considered as betraying the imperfection of the regularity of the lattice rearrangement, as the compression piece on being squeezed takes the shape of a cask.

Further it appears to be noteworthy that the symmetry of the pattern is imperfect in the strict sense in Figs. 19, 20, and 21, as the horizontal band is clearly visible on one side of the figure, but it is very faint or almost invisible on the other side. Several photographs were taken to find if there were any error in the experiment, but the test showed that the pattern in question was rightly due to the particular arrangement of the reflecting planes.

The meridian section examined with the monochromatic ray shows the six spots symmetrically arranged at equal distances from the centre, Fig. 10, while the non-central section parallel to a meridian plane, Fig. 11, gives also the six spots of which the middle ones are in the circumferential direction of the original piece. But the figure obtained from the cross section, Fig. 12, shows a circular ring with more or less faint spots of which the distribution is not so regular as the others. The film distance was again 61 mm in all these cases. The similarity of the patterns of Figs. 10 and 11, to that of Fig. 9 may be observed.

Thus the symmetrical pattern with the six spots seems to indicate that the atomic rearrangement takes place in such a

manner that the plane 2, 0, 0 of the elementary cube groups together in a certain condition to give rise to a vivid reflection of the incident ray, the two middle spots being in the direction of the expansion, but the rearrangement is probably not so thorough as in the case of the uniaxial extension (or the hard drawing), in which the reflection of other planes is also very clear.

Torsion.

The torsion specimens were taken also from a rolled copper rod after it had been annealed in just the same way as in the case of compression, and the strain was given by twisting the piece in a torsion testing machine until the breaking took place. The chief dimensions of the piece and the specific angle of torsion were as follows.

Specimen	Initial Diameter <i>mm</i>	Gauge Length <i>mm</i>	Extension %	Specific Angle of Torsion radians/cm
A	30.00	100	2.9	2.33
C	15.00	100	4.3	4.58

By the plastic torsion the material was found to expand in the axial direction of the piece, the amount of the extension after the breaking being as shown in the above table, the diameter being reduced at the same time. Here it is to be added that the axial expansion was almost free in the machine used for the present purpose, as the displacement in the said direction was resisted only by the rolling friction of the balls in a coupling of special design used for transmitting the applied torque to the hydraulic chamber (Messdose) with which the torque was measured.

Fig. 30.



Now the specimens taken for the examination were of the following kinds, Fig. 30.

- (i) Surface layer,
- (ii) Axial section,
- (iii) Cross section.

The surface layer cut parallel to the axis of the test piece was examined with the white ray, and the six banded pattern, Fig. 22, was obtained as in the case of compression. The central band about which the figure is symmetrical, stands nearly along the axial direction of the piece, i.e. the direction of the expansion, but not exactly.

To find the inclination of the central band to the axial direction of the piece, rectangular openings of narrow width were used instead of the round holes in the lead diaphragms, and the axial direction of the piece was put parallel to the longitudinal direction of the openings. By this means the angle between the central band and the image of the openings showed the required inclination. Thus facing the external surface of the specimen with the helical line running e.g. in the sense of a left handed screw, the direction of the central band is seen to be inclined clockwise by a few degrees from the axial direction.

The surface layer and the other sections being examined with the monochromatic ray, we find the usual pattern of the six spots in the cases of the surface layer, Fig. 23, and the axial section, Fig. 24, while the cross section, Fig. 25, shows that four of the six spots are arranged at an angle of 45° apart from the middle ones. The film distance was 61 mm as usual.

Thus the pattern of the figure obtained from the torsion specimens compared with that of compression shows a point of similarity namely that in both cases the six bands or spots are to be observed, owing to the reflection of the plane 2, 0, 0 of the elementary cube, and in both cases the symmetrical axis of the figure is in the direction in which the crystalline aggregate expands.

Characteristics of Interference Figures.

Recapitulating the experimental results reported in the previous sections, we see that the characteristic feature of the interference figure common to all the cases of the strained copper, as examined in the present test, is that the radiating bands or the spots (according as the incident ray is white or monochromatic) are symmetrically arranged about the two perpendicular directions. However, the difference between the pulled or hard drawn metal on one hand, and that compressed or twisted on the other, so far as the experiment shows, lies in the fact that in the former case the intensity of the central band or the corresponding spots is much stronger than the others, while such a difference of intensity is not observed in the latter case.

Moreover, in the former case the bands or the spots corresponding to the various planes of the lattice may be clearly distinguished,¹⁾ but the pattern in the latter case is chiefly due to the plane 2, 0, 0, mingled with intermediate bands or spots of weak intensity.

The angle which the normal to the last named plane makes with the axis of symmetry along the direction of extension is often in the neighbourhood of 55° , (corresponding to the case in which

1) Cp. the first report, p. 251.

the symmetrical axis coincides with the trigonal axis of the cubic lattice), but sometimes it is greater, being almost equal to 60° .

The drawn tubes taken for the examination may be classed as an intermediate product between the two groups above mentioned. Restricting our consideration to the case of the copper tube, we observe that the intensity of the central bands differs sensibly from the others, but the variety of the bands such as can be observed in hard drawn wires can not be observed. As pointed out in several places, the pattern of the drawn metal may depend on the axial extension and the lateral compression acting simultaneously during the process of drawing. If we assume that the effect of the pull is small, the pattern will take the form of compression, while it is reversed in the contrary case.

Returning to the main subject, the crystalline rearrangement brought into play by strains appears to be of such a kind that the atoms being displaced take the new position of equilibrium when the system is released from the external forces applied to it, and space lattice in the position of equilibrium shows a certain kind of regularity with respect to the direction of extension suffered by the aggregate of crystals.

The idea that the residual deformation of metals may be explained on the ground that the atomic movement takes place in a similar way to that in the case of magnetization due to the rotation of the elementary magnet¹⁾ was conventionally introduced by Ewing.²⁾ Such a conception may be considered to assume a more definite character than before as a result of the present examination.³⁾

1) J. A. Ewing, Magnetic Induction. Proc. Roy. Soc. A, 100, 1922, p. 449. Phil. Mag., 1922, p. 493. K. Honda and J. Okubo, Science Report, Sendai, First Series, V, 1916, p. 153.

2) Engineering, LXXXII, 1906, p. 203. Phil. Mag., 12, 1906, p. 254.

3) Cp. Encyklopädie Math. Wiss., IV, Art. 31, p. 767.



Fig. 21. Copper, compression, No. 10.
Meridian section.



Fig. 22. Copper, torsion, A.
Surface layer.



Fig. 24. Copper, torsion, C.
Axial section.



Fig. 23. Copper, torsion, C.
Surface layer.



Fig. 25. Copper, torsion, C.
Cross section.

Summary.

(1) The X-ray examination of drawn wires and tubes, and of copper specimens strained by extension, compression, and torsion, was performed, and the pattern of the interference figure obtained in each case was explained.

(2) First, the copper wire examined with the monochromatic ray showed the inference made in the first report with regard to the lattice rearrangement to be very probable.

(3) Next, the pattern obtained from the copper tube was not identical with that of the hard drawn wire, and it appeared like a wire in the early stage of drawing.

(4) Now in the case of the extension, the lattice planes as analysed in the case of hard drawn copper wires were all clearly observed, while in the case of the compression and torsion the pattern showed chiefly the rearrangement of a certain particular kind of plane as in the wire after a moderate degree of drawing.

(5) Thus the essential feature of the flow of copper under various kinds of deformation was to some extent revealed, and the similar and the dissimilar points were brought to light.

Fukuoka, May, 1922.



Fig. 1 Copper wire, A3



Fig. 2 Copper wire, A4

The direction of the unreflected ray is always normal to the plane of the figure.



Fig. 3 Copper, extension
Near gauge mark.



Fig. 4 Copper, extension
Near gauge mark.

The direction of the undiffracted ray is always normal to the plane of the figure.



Fig. 1. Copper wire, A1.



Fig. 2. Copper wire, A4.



Fig. 3. Copper, extension.
Near gauge mark.



Fig. 4. Copper, extension.
Near fractured section.



Fig. 5. Copper, extension.
Near fractured section.



Fig. 6. Steel tube.



Fig. 7. Copper tube.



Fig. 8. Brass tube



Fig. 9. Copper tube.



Fig. 10. Copper, compression, No. 6.
Meridian section.



Fig. 11. Copper, compression, No. 6.
Section parallel to a meridian plane.



Fig. 12. Copper, compression, No. 6.
Cross section.



Fig. 13. Copper, compression, No. 1.
Meridian section.



Fig. 14. Copper, compression, No. 1.
Meridian section. Near extremity.



Fig. 15. Copper, compression, No. 3.
Meridian section.



Fig. 16. Copper, compression, No. 4.
Meridian section.



Fig. 17. Copper, compression, No. 6.
Meridian section.



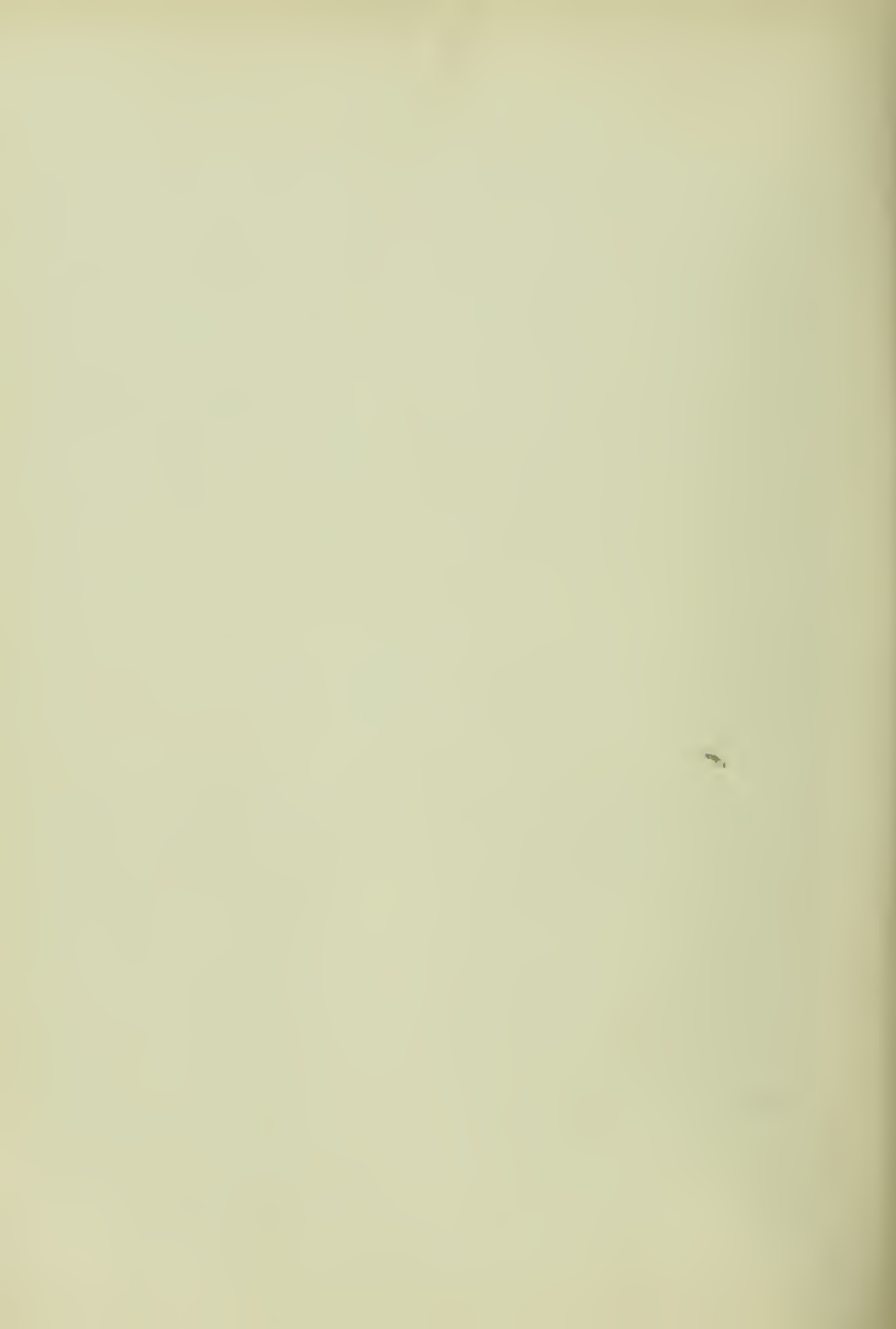
Fig. 18. Copper, compression, No. 6.
Meridian section. Near extremity.



Fig. 19. Copper, compression, No. 9.
Meridian section.



Fig. 20. Copper, compression, No. 9.
Meridian section. Not at centre.



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Zur Berechnung des beiderseits eingemauerten Trägers unter besonderer Berücksichtigung der Längskraft.

von

Fukuhei Takabeya, *Kogakushi.*

Einleitendes.

Bei der statischen Untersuchung eines eingemauerten Trägers setzt man ein vollkommen unnachgiebiges Einmauerungsmaterial voraus; die elastische Linie des Trägers besitzt danach an der Einmauerungsstelle eine unveränderliche Tangente.

Die Längskraft, mit der der Träger unter Umständen beansprucht werden kann, wird dabei gewöhnlich vollständig vernachlässigt. Es ist also unter einem eingemauerten Träger etwa ein solcher von der in Fig. 1 angedeuteten Konstruktion zu verstehen, in der eines der Trägerenden in wagerechter Richtung verschiebbar ist. Dieses Trägerende möge als „verschiebbar eingemauert“ bezeichnet werden.

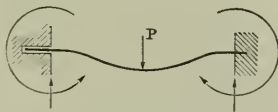


Fig. 1.

In der Praxis kommt bei derartigen Trägern gewöhnlich keine wagerechte Verschiebung in Frage, man sucht sogar eine solche durch konstruktive Ausbildung zu vermeiden. Ist der Träger lotrecht belastet und erfährt eine Durchbiegung, so existiert keine Nulllinie im Träger, deren Länge nach Formänderung unveränderlich bleibt. Der Träger wird also gleichzeitig mit dem Biegemoment auch noch durch die Längskraft beansprucht. Sie ist, wie einige Untersuchungen¹⁾ darüber zeigen, eine Zugkraft, die gewöhnlich auf die Beanspruchung des Trägers eine günstige Wirkung ausübt, indem sie eine Verminderung des Feldmomentes hervorruft.

Die statische Untersuchung des Trägers unter dieser Bedingung ist also eine in der Technik ziemlich häufig vorkommende Aufgabe; trotzdem scheint diese Frage bisher in der Literatur noch nicht behandelt worden zu sein. Im folgenden soll versucht werden, mit Hilfe der Elastizitätstheorie eine Lösung dieser Aufgabe zu finden; und zwar soll von der Voraussetzung gewisser Verschiebungen des Trägers an der Einmauerungsstelle ausgehend, der wirklich herrschende Gleichgewichtszustand unter dem die Längskraft zur Wirkung gelangen würde, bestimmt werden.

I. Entwicklung der allgemeinen Gleichungen.

§ 1. Vorbemerkungen. Um die allgemeine Gleichung zu entwickeln beginnen wir mit der Untersuchung eines frei aufliegenden Trägers.

1) F. Bleich, Die Theorie der nachgiebigen Tragsysteme, Der Eisenbau, 1916, S. 185.

K. Hayashi, Theorie des Trägers auf elastischer Unterlage, Berlin 1921, S. 197.

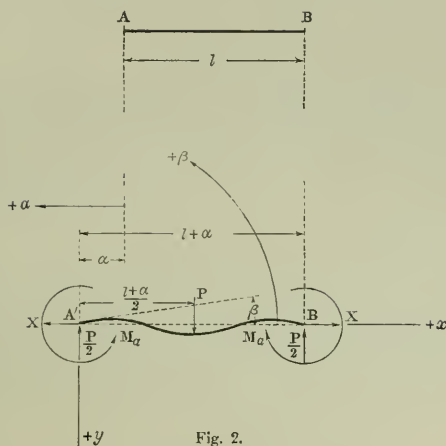


Fig. 2.

Der gerade Träger AB sei an beiden Enden unterstützt [Fig. 2]. Eine der Stützen, z. B. die an A sei wagerecht verschiebbar; diese Voraussetzung wird in der Festigkeitslehre gewöhnlich ausdrücklich nicht angegeben, wenigstens nicht solange es sich um einen geraden Träger handelt. Doch wollen wir hier darauf aufmerksam machen, denn es wird die wagerechte Verschiebung des Stützpunktes A wegen der elastischen Formänderung in der Folge eine wichtige Rolle spielen.

Als äussere Kräfte seien eine Einzellast P in der Mitte, eine wagerechte Kraft X am Ende A und ein Moment M_a an

beiden Enden angenommen. Der Träger nehme darunter eine neue Gleichgewichtsform $A'B$ an. Das Ende B bleibt dabei unbeweglich; in der Figur 2 ist der Deutlichkeit wegen die ganze Form $A'B$ gesondert dargestellt.

Es sei am linken Ende X nach links, M_a gegen den Sinn des Uhrzeigers positiv gerechnet. Die wagerechte Verschiebung a sowie die Verdrehung β des Trägerendes A soll in der positiven Richtung von bzw. X , M_a positiv gerechnet werden.

§ 2. Gleichung der elastischen Linie.

Erfährt ein Stabquerschnitt eine Biegung durch ein Moment M und nimmt die ursprünglich gerade Stabachse eine krumme Linie mit dem Krümmungshalbmesser ρ an, so gilt die bekannte Formel

$$\rho = \frac{EJ}{M}.$$

Kommt eine Beanspruchung durch eine Zugkraft N hinzu, so bleibt diese Grundgleichung bestehen, nur ist statt des Elastizitätsmasses E der Wert

$$(1) \quad K = E + \sigma_z = E + \frac{N}{F}$$

einzuführen¹⁾. F bedeutet die Querschnittsfläche des Stabes. Die Differentialgleichung der elastischen Linie lautet daher

1) M. Tolle, Die steife Kettenlinie. Zeitschr. d. V. D. I. 1897, S. 855-856.

$$\frac{d^2y}{dx^2} = -\frac{M}{\left[E + \frac{N}{F}\right]J} = -\frac{M}{KJ},$$

wenn die Ordinaten y lotrecht nach unten positiv gerechnet sind. Da bei den gebräuchlichen Spannungswerten der Wert $\sigma_s = \frac{N}{F}$ gegen E zu vernachlässigen ist, dürfte man an Stelle von K praktisch noch E setzen. Wählt man den Punkt A' als Koordinatennullpunkt, so ist

$$(2) \quad M = \frac{P}{2}x - M_a - Ky.$$

Somit nimmt die Differentialgleichung die lineare Form

$$\frac{d^2y}{dx^2} - \frac{K}{KJ}y + \frac{1}{KJ}\left[\frac{P}{2}x - M_a\right] = 0$$

an. Auch σ_s und also die Grösse K sind im allgemeinen mit X veränderlich. Da aber im vorliegenden Fall eine geringe Durchbiegung in Frage kommt, so setzen wir die Längskraft N und also σ_s über die ganze Trägerlänge als gleichbleibend voraus. Die Koeffizienten der Gleichung sind mithin konstant, und das allgemeine Integral lautet:

$$(3) \quad \left\{ \begin{array}{l} y = C_1 e^{\frac{2\omega x}{l+a}} + C_2 e^{-\frac{2\omega x}{l+a}} + \frac{P}{2K}x - \frac{M_a}{K}, \\ \text{wobei} \\ \omega = \frac{l+a}{2} \sqrt{\frac{K}{KJ}}. \end{array} \right.$$

Die Grösse ω ist eine reelle und imaginäre, natürliche Zahl, je nachdem X positiv oder negativ ist.

Die zwei Integrationsfestwerte C_1 , C_2 lassen sich folgendermassen bestimmen: Da am Ende A' , also für $x=0$ y verschwinden muss, ist

$$M_a = X [C_1 + C_2].$$

Der Symmetrie wegen muss für $x = \frac{l+a}{2}$ $\frac{dy}{dx} = 0$ sein; dies liefert

$$C_1 e^{\omega} - C_2 e^{-\omega} + \frac{P(l+a)}{4\omega X} = 0.$$

Aus diesen zwei Gleichungen erhält man:

$$\left\{ \begin{array}{l} C_1 = \frac{1}{2X \cos \omega} \left[M_a (\cos \omega - \sin \omega) - \frac{P(l+a)}{4\omega} \right] \\ C_2 = \frac{1}{2X \cos \omega} \left[M_a (\cos \omega + \sin \omega) + \frac{P(l+a)}{4\omega} \right]. \end{array} \right.$$

Es folgt:

$$(4) \quad y = \frac{P(l+a)}{4\omega X \cos \omega} \left[\frac{2\omega x}{l+a} \cos \omega - \sin \left(\frac{2\omega x}{l+a} \right) \right] \\ + \frac{M_a}{X \cos \omega} \left[\cos \left(\frac{2x}{l+a} - 1 \right) \omega - \cos \omega \right].$$

Die zweite Ableitung der Funktion lautet:

$$\frac{d^2 y}{dx^2} = \frac{M_a}{KJ} - \frac{\cos\left(\frac{2x}{l+a} - 1\right)\omega}{\cos \omega} - \frac{P\omega}{N(l+a)} \frac{\sin\left(\frac{2x\omega}{l+a}\right)}{\cos \omega}.$$

In bezug auf die ursprüngliche Differentialgleichung erhält man somit

$$(5) \quad M = \frac{P(l+a) \sin\left(\frac{2x\omega}{l+a}\right)}{4\omega \cos \omega} - \frac{M_a \cos\left(\frac{2x}{l+a} - 1\right)\omega}{\cos \omega}.$$

§ 3. Beziehungen zwischen den äusseren Kräften P , N , M_a und den Verschiebungen α , β . Die Verschiebungen α , β rühren von der elastischen Beschaffenheit des Trägers her. Sie können also durch die Anwendung des Prinzips der virtuellen Verrückungen festgestellt werden. Das Prinzip bezieht sich auf eine unendlich kleine Verrückung des Körpers. Wir setzen aber voraus, dass wir das Prinzip noch auf unsere Aufgabe anwenden müssen, in der die betreffende Verrückungen α , β sich auf den nicht als unendlich erfassbaren Wert belaufen. Es mögen dann zwischen α , β und den äusseren Kräften die bekannten Arbeitsgleichungen von der Form

$$(6) \quad \left\{ \begin{array}{l} 2 \int_0^{\frac{l+\alpha}{2}} \left[\frac{MM_1}{KJ} + \frac{NN_1}{KF} \right] dx = \alpha \\ \int_0^{\frac{l+\alpha}{2}} \left[\frac{MM_2}{KJ} + \frac{NN_2}{KF} \right] dx = \beta \end{array} \right.$$

als geltend betrachtet werden. P , N , M_a stecken in M , N . Die

virtuellen Grössen M_1 , N_1 und M_2 , N_2 mögen aus den in Fig. 3 *a* und *b* angegebenen Belastungszuständen entnommen werden.

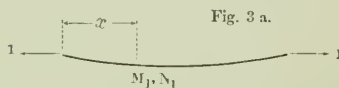


Fig. 3 a.

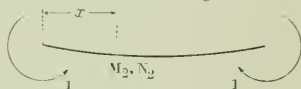


Fig. 3 b.

Wir befinden uns dabei in einem grossen Zweifel, welche Trägerform wir für die virtuellen Belastungen annehmen mögen.

Bei der statischen Berechnung von statisch unbestimmten Konstruktionen, wie Rahmen, Gewölben, die man mit Hilfe der Arbeitsgleichungen auszuführen pflegt, nimmt man für virtuelle Belastungen ein bestimmtes Hauptsystem an. Dieses denkt man sich in der Regel unabhängig von den äusseren Kräften, also von der erfolgten, deformierten Form der Konstruktionen. Im betreffenden Fall aber bedingt die eingetretene Durchbiegung des Trägers die Wirkung der äusseren Kraft X wesentlich oder unter Umständen gar ausschliesslich. Hätte man also als Trägerform

für virtuelle Belastungen entweder die ursprüngliche, gerade oder die deformierte, wenig gekrümmte angenommen, so würde man erkennen, dass die resultierenden virtuellen Grössen in beiden nicht völlig übereinstimmen.

Bei der letzteren Annahme beträgt $M_1=y$, während es bei der ersteren verschwindet.

Damit wir noch die angegebenen Arbeitsgleichungen in Verwendung bringen zu können, wollen wir in der Folge eine praktische Annäherung machen: Wir stellen uns für virtuelle Belastungen einen anfänglich gekrümmten Träger vor, dessen Achse etwa die mittlere der soeben erwähnten zwei Trägerformen besitzen möge. Es sei die Spannweite $l+\alpha$ und die Ordinaten halb so gross wie die wirklich auftretenden Durchbiegungen des in Rede stehenden Trägers [Fig. 4].

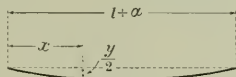


Fig. 4.

Es ist natürlich gar nicht einzusehen, aus welchem vernünftigen oder praktischen Grunde man solche Trägerform zugrunde legen will. Aus dem als Verallgemeinerung des Hookeschen Gesetzes aufzufassenden linearen Zusammenhang zwischen den Verschiebungen und Kräften in einer Konstruktion möge aber die angenommene als die zulässigste gestattet werden, wenigstens bis eine vernünftigere bewiesen wird.

Von der Zulässigkeit unserer Annahme wenigstens für praktischen Zweck werden wir uns späterhin noch auf einem anderen Wege überzeugen.

Es ist nun

$$(7) \quad \begin{cases} M_1 = -\frac{y}{2}, & N_1 = 1, \\ M_2 = -1, & N_2 = 0. \end{cases}$$

Da $N=X$, nimmt Gl. (6) die Gestalt

$$\begin{cases} \int_0^{\frac{l+\alpha}{2}} \left[\frac{M}{KF} y - \frac{2X}{KF} \right] dx = -\alpha \\ \int_0^{\frac{l+\alpha}{2}} \frac{M}{KF} dx = -\beta \end{cases}$$

an.

Führt man den Ausdruck für M aus Gl. (5) ein und wertet die Integralen aus¹⁾, so lautet:

1) Es mögen uns hierbei die Formeln

$$\begin{aligned} \int \sin kx dx &= \frac{\cos kx}{k} + c & \int \cos k \left(x - \frac{l}{2} \right) dx &= \frac{\sin k \left(x - \frac{l}{2} \right)}{k} + c \\ \int \sin^2 kx dx &= \frac{\sin 2kx}{4k} - \frac{x}{2} + c & \int \cos^2 k \left(x - \frac{l}{2} \right) dx &= \frac{\sin k (2x - l)}{4k} + \frac{x}{2} + c \\ \int \sin kx \cos k \left(x - \frac{l}{2} \right) dx &= \frac{\cos k \left(2x - \frac{l}{2} \right)}{4k} + \frac{x}{2} \sin \frac{kl}{2} + c \end{aligned}$$

$$(8) \left\{ \begin{aligned} & \frac{P^2}{16\lambda^2 \cos^2 \omega} \left[\frac{3 \sin 2\omega}{2\omega} - 2 - \cos 2\omega \right] + \frac{PM_a}{\lambda^2(l+a)} \frac{[\cos \omega - 1 - \frac{\omega}{2} \sin \omega]}{\cos \omega} \\ & + \frac{M_a^2 \left[1 - \frac{\sin 2\omega}{2\omega} \right]}{4\lambda K J \cos^2 \omega} + \frac{X}{KF} - \frac{a}{l+a} = 0, \\ & M_a = \frac{2\omega K J}{(l+a) \sin \omega} \left[\beta + \frac{P}{2\lambda} \sin \frac{\omega}{2} \sin \omega \right]. \end{aligned} \right.$$

Dies sind die Gleichungen, aus den X , M_a bei gegebenen α , β festgestellt werden. Eliminiert man darin M_a und das ausdrücklich angegebene λ , so ergibt sich für die Berechnung von X

$$(8_a) \left\{ \begin{aligned} & \frac{2 + \cos \omega - \frac{3 \sin \omega}{\omega}}{\omega^6 \cos^2 \frac{\omega}{2}} - \frac{8\beta K J [\sin \omega - \omega]}{P(l+a)^2 \omega^5 \cos^2 \frac{\omega}{2}} + \frac{32\beta^2 K^2 J^2 [\sin 2\omega - 2\omega]}{P^2(l+a)^4 \omega^3 \sin^2 \omega} \\ & - \frac{1024 K^2 J}{P^2(l+a)^6 P} + \frac{256 \alpha K^2 J^2}{P^2(l+a)^5 \omega^2} = 0. \end{aligned} \right.$$

Mit dem Wert M_a aus Gl. (8) nimmt Gl. (4) weiter die Form

$$\int x \sin kx dx = \frac{x \cos kx}{k} - \frac{\sin kx}{k^2} + c$$

$$\int x \cos k \left(x - \frac{l}{2} \right) dx = \frac{x \sin k \left(x - \frac{l}{2} \right)}{k} - \frac{\cos k \left(x - \frac{l}{2} \right)}{k^2} + c$$

gute Dienste leisten.

$$(4_a) \quad y = \frac{P}{2X} \left[x - \frac{(l+a) \Im \sin \left(\frac{x\omega}{l+a} \right) \Im \cos \left(\frac{\omega}{2} - \frac{\omega x}{l+a} \right)}{\omega \Im \cos \frac{\omega}{2}} \right] \\ + \frac{\beta(l+a) \Im \sin \left(\frac{\omega x}{l+a} \right) \Im \sin \left(\frac{x\omega}{l+a} - \omega \right)}{\omega \Im \sin \omega}$$

an.

Setzt man darin $x = \frac{l+a}{2}$, so erhält man

$$(9) \quad y_{\text{mitte}} = \frac{P(l+a)}{2\omega X} \left[\frac{\omega}{2} - \Im_g \frac{\omega}{2} \right] - \frac{\beta(l+a)}{2\omega} \Im_g \frac{\omega}{2}.$$

Für M ergibt sich

$$(5_a) \quad M = \frac{P(l+a)}{4\omega} \frac{\Im \sin \left(\frac{2\omega x}{l+a} - \frac{\omega}{2} \right)}{\Im \cos \frac{\omega}{2}} - \frac{2\beta\omega KJ \Im \cos \left(\omega - \frac{2\omega x}{l+a} \right)}{(l+a) \Im \sin \omega}.$$

Schliesslich, da $Q = \frac{\partial M}{\partial x}$, gilt

$$(10) \quad Q = \frac{P \Im \cos \left(\frac{2\omega x}{l+a} - \frac{\omega}{2} \right)}{2 \Im \cos \frac{\omega}{2}} + \frac{\beta X \Im \sin \left(\omega - \frac{2\omega x}{l+a} \right)}{\Im \sin \omega}.$$

§ 4. Sonderfälle. Es sei $M_a=0$, $X=0$. Dem Fall entspricht der sogenannte beiderseits unterstützte Träger mit einer Einzellast in der Mitte [Fig. 5]. Ueberhaupt setzt man dabei stillschweigend voraus, dass die Stützgelenke wagerecht vollkom-

men verschiebbar seien und redet danach nicht von einer Längskraft, welche, bei unvollkommen verschiebbaren Stütz-
gelenke, infolge der Längenänderung des Trägers, wenn auch
sehr klein, im Träger entstehen könnte.

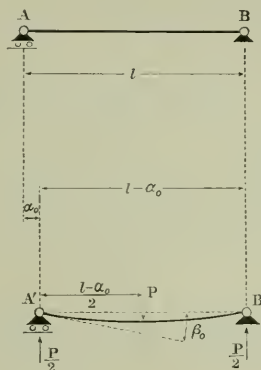


Fig. 5.

Gl. (8) liefert für die Verschiebungen am Ende A' :

$$(11) \quad \left\{ \begin{array}{l} a_0 = \frac{P^2 l^5}{960 E^2 J^2} \left[1 - \frac{a_0}{l} \right]^5 \cong \frac{P^2 l^5}{960 E^2 J^2} \\ \beta_0 = \frac{P l^2}{16 E J} \left[1 - \frac{a_0}{l} \right]^2 \cong \frac{P l^2}{16 E J} \end{array} \right.$$

Wie man leicht beweist, finden diese stets wie in der Fig. 5
statt. Ferner erhält man aus Gln. (4), (5), indem man darin

$\alpha = -\alpha_0$ setzt :

$$(12) \left\{ \begin{aligned} y &= \lim_{\substack{X=0 \\ M_a=0 \\ \alpha=-\alpha_0}} \left[\frac{P(l+a)}{4\omega X \cos \omega} \left\{ \frac{2\omega x}{l+a} \cos \omega - \sin \left(\frac{2\omega x}{l+a} \right) \right\} \right. \\ &\quad \left. + \frac{M_a}{X \cos \omega} \left\{ \cos \left(\frac{2x\omega}{l+a} - \omega \right) - \cos \omega \right\} \right] \\ &= \frac{P}{16EJ} \left[\gamma^2 \left\{ 1 - \frac{P^2 \gamma^4}{960 E^2 J^2} \right\}^2 - \frac{4}{3} \gamma^3 \right], \\ M &= \lim_{\substack{X=0 \\ M_a=0 \\ \alpha=-\alpha_0}} \left[\frac{P(l+a)}{4\omega} - \frac{\sin \left(\frac{2x\omega}{l+a} \right)}{\cos \omega} - \frac{M_a \cos \left(\frac{2x\omega}{l+a} - \omega \right)}{\cos \omega} \right] = \frac{P}{2} x. \end{aligned} \right.$$

Die Formeln sind aus den Lehrbüchern wohl bekannt. Daraus folgt

$$(13) \left\{ \begin{aligned} y_{mitte} &= \frac{P(l-\alpha_0)^3}{48EJ} = \frac{Pl^3}{48EJ} \left[1 - \frac{P^2 \gamma^4}{960 E^2 J^2} \right]^3 \\ M_{mitte} &= \frac{P(l-\alpha_0)}{4} = \frac{Pl}{4} \left[1 - \frac{P^2 \gamma^4}{960 E^2 J^2} \right]. \end{aligned} \right.$$

An zweiter Stelle sei $\beta=0$, $X=0$. Beschränkt man sich auf eine einzige äussere Kraft P , so ist dabei der in den Lehrbüchern sogenannte beiderseits eingespannte Träger angedeutet, bei dessen Berechnung man ein einseitig wagerecht vollkommen verschiebbares Einmauerungsende annimmt, ohne darauf besonders aufmerksam zu machen [Fig. 6]. Man hat danach für α , M_a

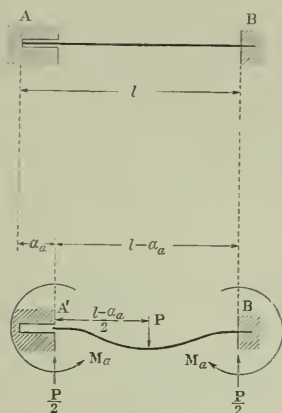


Fig. 6.

$$(14) \left\{ \begin{array}{l} a_a \approx \frac{Pl^5}{15360E^2J^2}, \\ M_a \approx \frac{Pl}{8}. \end{array} \right.$$

Die Verschiebung a_a findet ebenfalls nach rechts statt. Setzt man $a = -a_a$ in Gl. (4_a) (δ_a), so erhält man

$$\left. \begin{array}{l} y = \lim_{\substack{x \rightarrow 0 \\ \beta \rightarrow 0 \\ a \rightarrow a_a}} \left[\frac{P}{2X} \left\{ x - \frac{\beta(l+a) \sin \frac{x\omega}{l+a} \cos \left(\frac{\omega}{2} - \frac{x\omega}{l+a} \right)}{\omega \cos \frac{\omega}{2}} \right\} \right. \\ \left. + \frac{\beta(l+a) \sin \left(\frac{x\omega}{l+a} \right) \sin \left(\frac{\omega x}{l+a} - \omega \right)}{\omega \sin \omega} \right] \end{array} \right\}$$

$$(15) \left\{ \begin{aligned} &= \frac{P}{16EJ} \left[x^2(l-a) - \frac{4}{3}x^3 \right], \\ M &= \lim_{\left(\begin{smallmatrix} x=0 \\ \beta=0 \\ a=a_a \end{smallmatrix} \right)} \left[\frac{P(l+a)}{4\omega} \frac{\Im \ln \left(\frac{2x\omega}{l+a} - \frac{\omega}{2} \right)}{\Im \cos \frac{\omega}{2}} - \frac{2\beta\omega KJ \Im \cos \left(\omega - \frac{2\omega x}{l+a} \right)}{(l+a) \Im \ln \omega} \right] \\ &= \frac{P}{2}x - \frac{P(l-a)}{8} \end{aligned} \right.$$

und daher

$$(16) \left\{ \begin{aligned} y_{mitte} &= \frac{Pl^3}{192EJ} \left[1 - \frac{P^2 l^4}{15360 E^2 J^2} \right]^3, \\ M_{mitte} &= \frac{Pl}{8} \left[1 - \frac{P^2 l^4}{15360 E^2 J^2} \right]. \end{aligned} \right.$$

Als Beispiel nimmt man einen Doppel- T -Träger mit

$$J=170 \text{ cm}^4; F=10,6 \text{ cm}^2;$$

$$E=2100000 \text{ kg/cm}^2; l=400 \text{ cm};$$

$$P=100 \text{ kg}$$

an.

Die Berechnung liefert :

	Falls $X=0, \beta=0$	Falls $X=0, M_a=0$
a	0,000052 cm	0,00084 cm
β	0	0,00280
y_{mitte}	0,093371 cm	0,37348 cm
M_{mitte}	5000,00 cmkg	10000,00 cmkg
M_a	5000,00 cmkg	0

§ 5. Bemerkungen. Zur Prüfung der Genauigkeit der vorhergehenden Theorie soll in Weiteres hinzugefügt werden.

Die Verschiebung α_0 in Fig. 5 sei auf einem anderen Wege zu ermitteln. Unter der Annahme der Unveränderlichkeit der Trägerlänge nach der Formänderung erhält man

$$l = 2 \int_0^{\frac{l-\alpha_0}{2}} \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx.$$

Für geringe Durchbiegungen und daher kleine Werte von $\frac{dy}{dx}$ ergibt sich also annähernd:

$$l \cong 2 \int_0^{\frac{l-\alpha_0}{2}} \left[1 + \frac{1}{2} \left(\frac{dy}{dx} \right)^2 \right] dx.$$

Der Ausdruck für $\frac{dy}{dx}$ ist aus der Gleichung

$$y = \frac{P(l-\alpha_0)^3}{16EJ} \left[\frac{x}{l-\alpha_0} - \frac{4}{3} \frac{x^3}{(l-\alpha_0)^3} \right]$$

abzuleiten. Die Auswertung des Integrals liefert

$$l \cong l - \alpha_0 + \frac{P^2 l^5}{960 E^2 J^2}$$

und folglich

$$\alpha_0 \cong \frac{P^2 l^5}{960 E^2 J^2},$$

einen Ausdruck, welcher völlig mit dem in Gl. (11) übereinstimmt.

Unter analoger Erwägung gelangt man für den allgemeinen Fall [s. Fig. 2] zu der Gleichung

$$\frac{P^2}{16X^2 \cos^2 \omega} - \frac{3 \sin 2\omega}{2\omega} - 2 - \cos 2\omega + \frac{PM_a}{X^2(l+a)} \left[\cos \omega - 1 - \frac{\omega}{2} \operatorname{tg} \omega \right] \\ + \frac{M_a^2 \left[1 - \frac{\sin 2\omega}{2\omega} \right]}{4XKJ \cos^2 \omega} + \frac{X}{KF} - \frac{a}{l+a} = 0.$$

Man vergleiche mit Gl. (8).

Ich habe ferner für einige andere Aufgaben dieselbe Ueberzeugung gewonnen. Die Ergebnisse sollen hier angegeben werden :

Für die Fälle in Fig. 7, 8 liefert die Arbeitsgleichung :

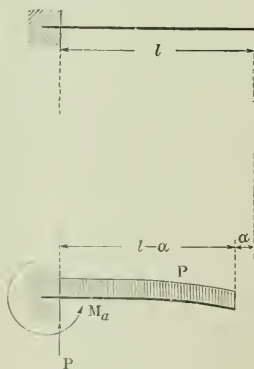


Fig. 7.

bzw. :

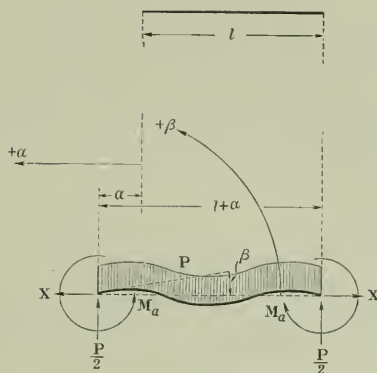


Fig. 8.

$$\alpha = \int_0^{l-\alpha} \frac{MM_1}{EJ} dx = \int_0^{l-\alpha} \frac{M_1 y}{2EJ} dx \approx \frac{P^2 l^5}{112 E^2 J^2},$$

$$\begin{aligned} \alpha = \int_0^{l+\alpha} \left[\frac{MM_1}{KJ} + \frac{NN_1}{KF} \right] dx &= \frac{P^2(l+\alpha)}{96X^2 \sin^2 \omega} \left[6 + \frac{9 \sin 2\omega}{\omega} - \frac{24 \sin^2 \omega}{\omega^2} - 4 \sin^2 \omega \right] \\ &+ \frac{\beta^2(l+\alpha)}{8 \sin^2 \omega} \left[2 - \frac{\sin 2\omega}{\omega} \right] + \frac{P(l+\alpha)\beta}{8X \sin^2 \omega} \left[2 + \frac{\sin 2\omega}{\omega} - 4 \frac{\sin^2 \omega}{\omega^2} \right] \\ &+ \frac{X(l+\alpha)}{KF}. \end{aligned}$$

Anderseits geben die Beziehungen

$$\left\{ \begin{array}{ll} a = \int_0^{l-a} \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx - (l-a) & \text{in Fig. 7,} \\ a = (l+a) - \int_0^{l+a} \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx + \frac{\lambda(l+a)}{KF} & \text{in Fig. 8,} \end{array} \right.$$

zwei mit den entsprechenden beiden vorangehenden Gleichungen völlig übereinstimmende Ausdrücke für a .

II. Der Träger ist vollkommen eingespannt und trägt nur eine Einzellast in der Mitte [Fig. 9].

§ 6. Aufstellung der Gleichungen. Unter „vollkommen eingespannt“ versteht man die Befestigungsart eines Trägers, in der der Träger weder einer Verschiebung noch einer Verdrehung unterworfen ist.

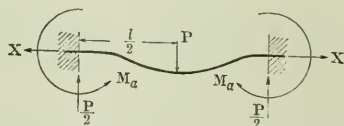


Fig. 9.

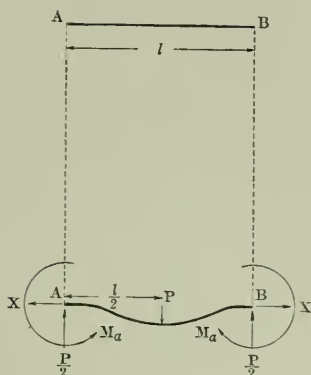


Fig. 10.

Dem vorgelegten Fall entspricht also der im vorigen Abschnitt behandelte Träger [Fig. 10], wenn man die Bedingung

$$\begin{cases} \alpha=0 \\ \beta=0 \end{cases}$$

beifügt. Das Moment M_a sowie die Kraft X , die in § 1 als äussere Ursache bezeichnet wurden, treten jetzt bei dem in Frage stehenden Träger als innere Ursache auf und zwar als eine solche, die er infolge der Formänderung wegen P und unter der angegebenen Bedingung in sich hervorruft.

Die X -Gleichung (8) rechnet sich um auf

$$(17) \quad \frac{2 + \cos \omega - \frac{3 \sin \omega}{\omega}}{\omega^6 \cos^2 \frac{\omega}{2}} = \frac{1024 K^2 J^3}{P^2 l^6 P}.$$

Die Gleichung ist in hohem Grade transzendent; man könnte aus ihr keinen praktischen Schluss ziehen. Das Vorzeichen von X lässt sich indessen durch die folgende Überlegung feststellen: Es handle sich um einen positiven Wert von X . Die Zahl $\omega = \frac{l}{2} \sqrt{\frac{X}{KJ}}$ ist dann reell und die Funktion

$$\frac{2 + \cos \omega - \frac{3 \sin \omega}{\omega}}{\omega^6 \cos^2 \frac{\omega}{2}}$$

bleibt stets positiv. Sie nimmt, wie die Tabelle 1 zeigt, mit ω ab. Wenn ω imaginär ist, was bei einem negativen X geschieht, ist die Funktion negativ und nimmt mit ω zu. Es folgt daraus, dass die letzte Gleichung keine negative Wurzel X gestattet, denn die rechte Seite derselben stellt für ein positives ebenso wie ein negatives P stets eine positive Zahl dar. Die Kraft X ist also ein Zug, gleichgültig gegen das Vorzeichen von P .

Die letzterwähnte Funktion ist noch in Fig. 11 graphisch dargestellt. Es sind dabei die Funktionswerte für die imaginären ω eingetragen.

$$\frac{2 + \cos \omega - \frac{3 \sin \omega}{\omega}}{\omega^5 \cos^2 \frac{\omega}{2}}$$

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$$\frac{2 \cdot \cos \omega - \frac{3 \sin \omega}{\omega}}{\omega \cdot \cos^2 \frac{\omega}{2}}$$

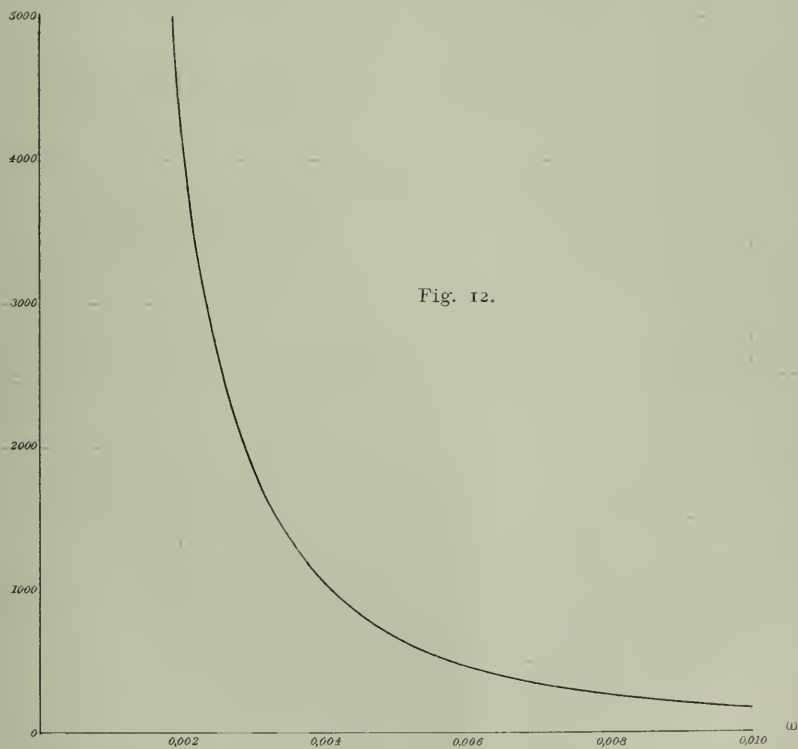


Fig. 12.

Tabelle 1.

ω	$\frac{2 + \zeta \cos \omega - 3 \zeta \sin \omega}{\omega^6 \zeta \cos^2 \frac{\omega}{2}}$	ω	$\frac{2 + \zeta \cos \omega - 3 \zeta \sin \omega}{\omega^6 \zeta \cos^2 \frac{\omega}{2}}$
0	∞		
0,001	16666,66329	0,051	6,40442
2	4166,66329	52	6,16034
3	1851,84847	53	5,92994
4	1041,66329	54	5,71222
5	666,66329	55	5,50627
6	462,95959	56	5,31126
7	340,13263	57	5,12641
8	260,41329	58	4,95105
9	205,75794	59	4,78453
0,010	166,66329	0,060	4,62626
11	137,73768	61	4,47571
12	115,73737	62	4,33239
13	98,61566	63	4,19584
14	85,03064	64	4,06564
15	74,07070	65	3,94140
16	65,10079	66	3,82277
17	57,66676	67	3,70953
18	51,43696	68	3,60101
19	46,16468	69	3,49729
0,020	41,66329	0,070	3,39799
21	37,78953	71	3,30285
22	34,43189	72	3,21165
23	31,50261	73	3,12417
24	28,93181	74	3,04021
25	26,66329	75	2,95959
26	24,65146	76	2,88213
27	22,85899	77	2,80767
28	21,25513	78	2,73606
29	19,81431	79	2,66714
0,030	18,51515	0,080	2,60080
31	17,33968	81	2,53689
32	16,27267	82	2,47531
33	15,30119	83	2,41595
34	14,41416	84	2,35869
35	13,60207	85	2,30344
36	12,85671	86	2,25010
37	12,17096	87	2,19859
38	11,53864	88	2,14883
39	10,95423	89	2,10074
0,040	10,41329	0,090	2,05424
41	9,91136	91	2,00927
42	9,44485	92	1,96575
43	9,01051	93	1,92364
44	8,60544	94	1,88256
45	8,22708	95	1,84335
46	7,87312	96	1,80508
47	7,54152	97	1,76799
48	7,23042	98	1,73202
49	6,93818	99	1,69714
50	6,66329	0,100	1,66330
		∞	0

Die Gleichung für die elastische Linie lautet:

$$(18) \quad y = \frac{P}{2\lambda X} \left[x - \frac{l \sin\left(\frac{x\omega}{l}\right) \cos\left(\frac{\omega}{2} - \frac{\omega x}{l}\right)}{\omega \cos \frac{\omega}{2}} \right].$$

Setzt man darin $x = \frac{l}{2}$, so ergibt sich

$$(19) \quad y_{mitte} = \frac{Pl}{2\omega \lambda X} \left[\frac{\omega}{2} - \operatorname{tg} \frac{\omega}{2} \right].$$

Gln. (5_e), (10) liefern

$$(20) \quad \left\{ \begin{array}{l} M = \frac{Pl}{4\omega} \frac{\sin\left(\frac{2\omega x}{l} - \frac{\omega}{2}\right)}{\cos \frac{\omega}{2}}, \\ M_a = \frac{Pl}{4\omega} \operatorname{tg} \frac{\omega}{2}, \\ Q = \frac{P \cos\left(\frac{2\omega x}{l} - \frac{\omega}{2}\right)}{2 \cos \frac{\omega}{2}}. \end{array} \right.$$

§ 7. Ergänzungsbeispiel. Es soll der in § 4 angegebene Träger unter vollkommener Einmauerung berechnet werden.

Es ist

$$\begin{aligned} K &= 2100000 + \frac{X}{10,6} \quad [Gl. (1)] \\ \omega &= \frac{400}{2} \left[\frac{X}{170 \left(2100000 + \frac{X}{10,6} \right)} \right]^{\frac{1}{2}} \\ &\cong \frac{\sqrt{X}}{94,4722 \left(1 + \frac{X}{10^7 \cdot 4,452} \right)}. \end{aligned}$$

X Kg.

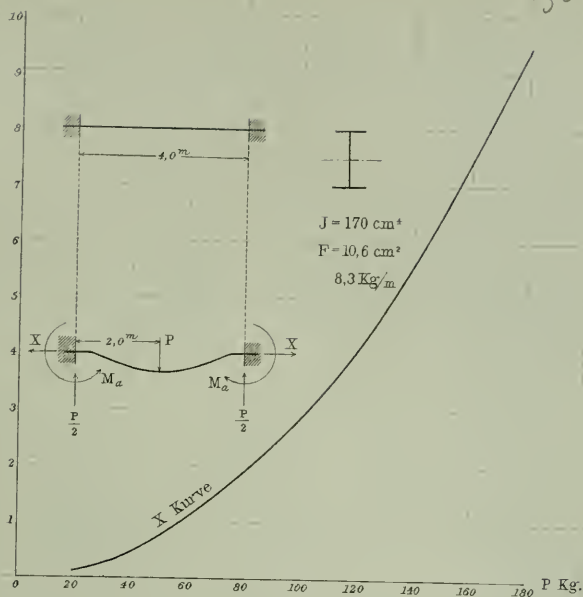


Fig. 13.

X Kg.

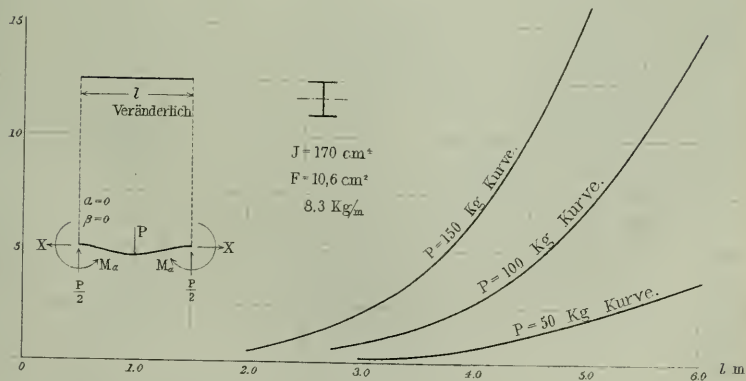


Fig. 14.

Somit lässt sich die Kraft X mittels der Gl. (17) durch Probieren berechnen. Man findet

$$X=2,9 \text{ kg}$$

und ferner

$$y_{\text{mitte}}=0,093368 \text{ cm}$$

$$M_{\text{mitte}}=4999,86 \text{ cmkg.}$$

In bezug auf S. 294 ist

$$\frac{0,093368}{0,093371}=0,999968 \text{ für } y_{\text{mitte}}$$

$$\frac{4999,86}{5000,00}=0,999972 \text{ für } M_{\text{mitte}}.$$

Die Durchbiegung sowie das Biegemoment in der Trägermitte sind also unter Berücksichtigung der Längskraft kleiner als bei dem Träger mit den wagerecht verschiebbaren Einspannungsenden. Im allgemeinen kann man die anfangs in der Arbeit erwähnte Verminderung des Feldmomentes rechtfertigen.

Aus den Rechnungsergebnisse bemerkt man, dass der Einfluss von Kraft X auf y_{mitte} M_{mitte} äusserst gering ist. Die Kraft X selbst aber beläuft sich auf etwa 3 v. H. von P . Bei der Bestimmung der wirklichen Spannungen im Träger möge sie vernachlässigt werden. Dies kann aber bei nachgiebigem Einmauerungsmaterial nie der Fall sein; davon werden wir uns in den folgenden Paragraphen vielfach überzeugen können.

In Fig. 13 ist X als Funktion von P aufgetragen. Man sieht darauf, dass die Zunahme von X mit P wächst.

An zweiter Stelle sei bei dem angegebenen Träger die Licht-

Die Einspannung eines Trägers an einem eingemauerten Ende, wenn das Einmauerungsmaterial nachgiebig ist, ist von Prof. K. Hayashi¹⁾ so definiert, dass der Träger daselbst durch ein negatives Biegemoment beansprucht wird, indem man sich auf von oben nach unten wirkende Lasten beschränkt. Von der wagerechten Verschiebung des Trägerendes, der der Träger bei diesen Belastungen noch mit der Mauer oder einzeln unterliegen kann [vgl. Fig. 15], ist bei Hayashi nicht die Rede. Zur schärferen Definition des Bezeichnungen Einspannung, möge also solche hinzugefügt werden, dass der beiderseits eingemauerte Träger mit lotrechter Last als äussere Kraft, auch wenn das Endmoment Null ist, noch als an den Enden eingespannt angesehen werden soll, solange das Einmauerungsmaterial eine Zugkraft an den Enden des Trägers auszuüben vermag.

Nach diesen Voraussetzungen können verschiedene Fälle von eingespannten Trägern unterschieden werden. X , M_a könnten dabei im allgemeinen entweder als äussere oder als innere Ursache betrachtet werden. Im vorliegenden Abschnitte wollen wir uns aber auf die Annahme beschränken, dass der Träger nur eine einzige äussere Kraft P trägt und also X , M_a nur als die von P herrührenden inneren Kräfte anzusehen sind.

§ 9. Der Träger ist beiderseits mit wagerecht unschiebbaren Gelenken versehen. Dem Fall entspricht $\alpha=0$, $M_a=0$ [Fig. 16].

1) K. Hayashi, Theorie des Trägers auf elastischer Unterlage, Berlin 1921, S. 180.

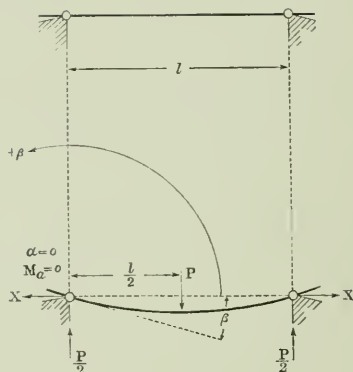


Fig. 16.

Für X erhält man aus Gl. (8), in der X mit $\left(\frac{2\omega}{l}\right)^2 KJ$ ersetzt wird:

$$(21) \quad \frac{2 + \cos 2\omega - \frac{3 \sin 2\omega}{2\omega}}{(2\omega)^6 \cos^2 \omega} = \frac{16K^2 J^3}{P^2 l^6 F'}.$$

Man sieht wieder ohne weiteres, dass am Trägerende eine Zugkraft herrscht.

Die Funktion

$$\frac{2 + \cos 2\omega - \frac{3 \sin 2\omega}{2\omega}}{(2\omega)^6 \cos^2 \omega}$$

ist in der Tabelle 2 zahlenmässig angegeben.

Setzt man $M_a=0$ in Gl. (8), so lautet die Verdrehung β

$$(22) \quad \beta_1 = -\frac{P}{2N} \left[1 - \frac{1}{\cos \omega} \right].$$

Das Minuszeichen zeigt, dass die Verdrehung im Sinne des Uhrzeigers stattfindet [Fig. 16].

Ferner folgt:

$$(23) \quad \left\{ \begin{array}{l} y_{\text{mitte}} = \frac{Pl}{4N} \left[1 - \frac{\mathfrak{T}g\omega}{\omega} \right], \\ M_{\text{mitte}} = \frac{Pl}{4\omega} \mathfrak{T}g\omega. \end{array} \right.$$

Berechnet man nach der hier angegebenen Bedingung in § 4 behandelte Träger, so erhält man als die X -Kurve für das veränderliche P eine parabolförmige Kurve [Fig. 17].

Tabelle 2.

ω	$\frac{2 + \zeta \cos 2\omega - \frac{3}{2} \zeta \sin 2\omega}{(2\omega)^6 \zeta \cos^2 \omega}$	ω	$\frac{2 + \zeta \cos 2\omega - \frac{3}{2} \zeta \sin 2\omega}{(2\omega)^6 \zeta \cos^2 \omega}$
0	∞	0,051	1,59858
0,001	4166,66329	52	1,53756
2	1041,66329	53	1,47996
3	462,95959	54	1,42553
4	260,41329		
5	166,66329	55	1,37404
6	115,77737	56	1,32529
7	85,03064	57	1,27908
8	65,10079	58	1,23524
9	51,43696	59	1,19361
0,010	41,66329	0,060	1,15404
11	34,43189	61	1,11641
12	28,93181	62	1,08058
13	24,95146	63	1,04644
14	21,25513	64	1,01389
15	18,51515	65	0,98283
16	16,27267	66	0,95317
17	14,41416	67	0,92483
18	12,85671	68	0,89773
19	11,53864	69	0,87180
0,020	10,41379	0,070	0,84698
21	9,44485	71	0,82319
22	8,60544	72	0,80039
23	7,87712	73	0,77852
24	7,23042	74	0,75753
25	6,66329	75	0,73738
26	6,16034	76	0,71801
27	5,71222	77	0,69940
28	5,31126	78	0,68150
29	4,95105	79	0,66427
0,030	4,62626	0,080	0,64768
31	4,33239	81	0,63171
32	4,06564	82	0,61631
33	3,82277	83	0,60147
34	3,60101	84	0,58716
35	3,39799	85	0,57334
36	3,21165	86	0,56001
37	3,04021	87	0,54713
38	2,88213	88	0,53469
39	2,73606	89	0,52267
0,040	2,60080	0,090	0,51105
41	2,47531	91	0,49980
42	2,35869	92	0,48893
43	2,25010	93	0,47840
44	2,14883	94	0,46820
45	2,05424	95	0,45833
46	1,96575	96	0,44876
47	1,88286	97	0,43943
48	1,80508	98	0,43049
49	1,73202	99	0,42177
50	1,66330	0,100	0,41331
		∞	0

308a

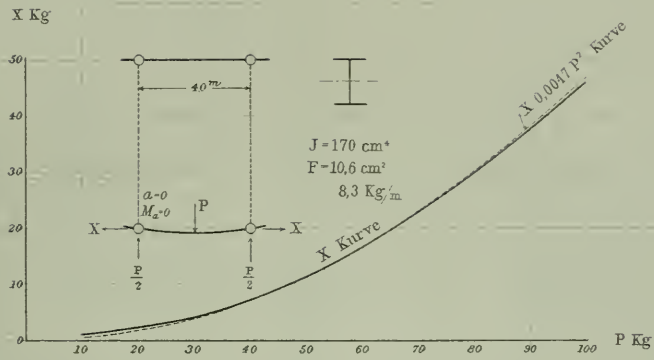


Fig. 17.

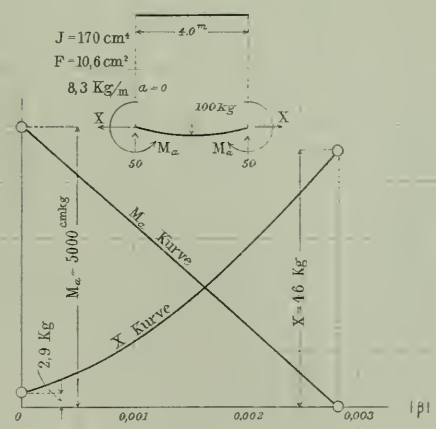


Fig. 18.

Die Kurve möge mit hinreichender Genauigkeit durch

$$X=0,0047 P^2$$

angegeben werden. Darnach verändert sich die Zugkraft X mit P^2 . In der Figur ist die letztere Kurve mit gestrichelter Linie eingezeichnet.

Ist $P=100$ kg, so berechnet sich

$$X=46 \text{ kg,}$$

$$\beta=0,0027978,$$

$$y_{\text{mitte}}=0,37280 \text{ cm,}$$

$$M_{\text{mitte}}=9982,87 \text{ cmkg.}$$

X ist etwa halb so gross wie P . Der Träger ist also durch eine verhältnissmässig grosse Längskraft beansprucht. Die grösste Spannung in der Trägermitte berechnet sich zu

$$\begin{aligned}\sigma &= \frac{M_{\text{mitte}}}{W} + \frac{X}{F} \\ &= \frac{9982,87}{34} + \frac{46}{10,6} \\ &= 297,95 \text{ kg/cm}^2.\end{aligned}$$

Bei der Berechnung derartiger Träger pflegt man unter der Annahme wagerecht verschiebbarer Enden die Kraft X völlig ausser Acht zu lassen. In bezug auf §4 lautet σ danach

$$\sigma = \frac{5000}{34} = 147,06 \text{ kg/cm}^2.$$

Man sieht also, dass unter Voraussetzung wagerecht nicht ver-

schiebbarer Enden die Nachgiebigkeit der Mauer gegen Verdrehung eine ungefähr doppelte Vergrößerung der massgebenden Spannungsgrösse zur Folge hat.

§ 10. Die Mauer ist gegen wagerechten Zug vollkommen fest, doch sei sie gegen Verdrehung in gewissem Grade nachgiebig [Fig. 19].

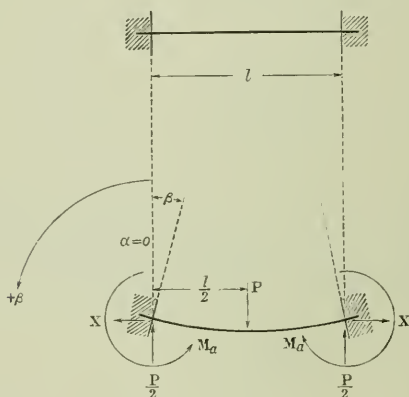


Fig. 19.

Es ist $\alpha=0$. Die Verdrehung β ist nicht mehr gleich β_1 [Gl. (22)], sondern fällt absolut kleiner aus als $|\beta_1|$ und zwar nimmt $|\beta|$ mit der Widerstandsfähigkeit des Einmauerungsmaterials ab, bis sie bei einem gegen Verdrehung vollkommen unnachgiebigen Einmauerungsmaterial einen Nullwert besitzt.

Das Endmoment M_a wächst dabei im Gegenteil von Null an und erhält schliesslich einen bestimmten Wert; dies ist der

Grenzwert, der den äussersten Widerstand ausdrückt, den die Mauer zu erweisen vermag. $\beta=0$ ist also der untere Grenzwert der Verdrehung, denn eine weitere Abnahme von β , was eine Annahme desselben im positiven Betrag bedeutet, würde nur dann möglich sein, wenn ein äusseres Moment zu dem soeben genannten Wert von M_a hinzugefügt würde.

Setzt man in Gl. (8.) $\alpha=0$, so gilt für die X -Gleichung

$$(24) \left\{ \begin{aligned} & \frac{2 + \cos \omega - \frac{3 \sin \omega}{\omega}}{\omega^6 \cos^2 \frac{\omega}{2}} - \frac{8\beta KJ [\sin \omega - \omega]}{Pl^2 \omega^5 \cos^2 \frac{\omega}{2}} \\ & + \frac{32\beta^2 K^2 J^2 [\sin 2\omega - 2\omega]}{P^2 l^4 \omega^3 \sin^2 \omega} - \frac{1024 K^2 J^3}{l^2 l^6 P^2} = 0. \end{aligned} \right.$$

Ferner folgt aus Gln. (9), (5_a)

$$(25) \left\{ \begin{aligned} y_{mitte} &= \frac{Pl}{2\omega X} \left[\frac{\omega}{2} - \tan^2 \frac{\omega}{2} \right] - \frac{\beta l}{2\omega} \tan^2 \frac{\omega}{2}, \\ M_{mitte} &= \frac{Pl}{4\omega} \tan^2 \frac{\omega}{2} - \frac{2\beta \omega KJ}{l \sin \omega}. \end{aligned} \right.$$

Zur näheren Erklärung berechnet man den Träger in §4 unter der vorgelegten Annahme. Die Rechnungsergebnisse sind in Fig. 18 aufgetragen.

Die Längskraft X nimmt also mit $|\beta|$ verhältnismässig rasch zu. Das Endmoment M_a nimmt mit wachsenden X ungefähr linearisch ab.

Ist $\beta = -0,001$, so berechnet sich

$$X = 12 \text{ kg}$$

und

$$y_{\text{mitte}} = 0,19332 \text{ cm}$$

$$M_{\text{mitte}} = 6784,04 \text{ cmkg.}$$

In bezug auf §4 ist

$$\left. \begin{array}{l} 0,19332 \\ 0,093371 \end{array} \right\} = 2,070450 \quad \text{für } y_{\text{mitte}}$$

$$\left. \begin{array}{l} 0,19332 \\ 0,37348 \end{array} \right\} = 0,517618$$

$$\left. \begin{array}{l} \frac{6784,04}{5000,00} \\ \frac{6784,04}{10000,00} \end{array} \right\} = 1,356808 \quad \text{für } M_{\text{mitte.}}$$

$$\left. \begin{array}{l} \frac{6784,04}{10000,00} \end{array} \right\} = 0,678404$$

Bei dem angenommenen β fällt also in der Trägermitte die Durchbiegung etwa zweimal, das Moment anderthälbmal so gross aus, wie der Träger beiderseits mit wagerecht verschiebbaren Enden versehen ist. In Vergleich zu dem Fall, wo der Träger mit wagerecht verschiebbaren Gelenke befestigt ist, vermindern sich diese Grössen mit grober Annäherung auf $\frac{1}{2}$.

§ 11. Die Mauer ist gegen Verdrehung des Trägerendes vollkommen unnachgiebig, gestattet aber eine gewisse wagerechte Verschiebung desselben [Fig. 20].

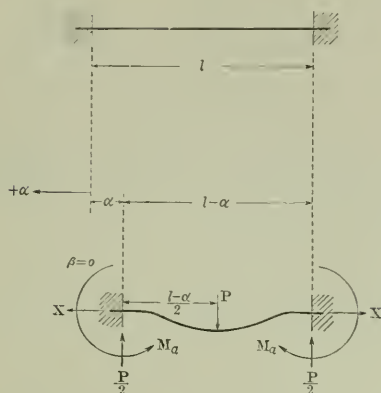


Fig. 20.

Das zu Anfang von §10 für β Gesagte gilt hierbei für α . Es ist $\beta=0$. Gl. (8_a) liefert für X

$$(26) \quad 2 + \cos \omega - \frac{3 \sin \omega}{\omega} + \frac{256 a K^2 J^2}{P^2 \omega^2 (l + a)^5} - \frac{1024 K^2 J^3}{P^2 l (l + a)^6} = 0.$$

Ferner erhält man

$$(27) \quad \left\{ \begin{array}{l} y_{\text{mitte}} = \frac{P(l+a)}{2\omega X} \left[\frac{\omega}{2} - \mathfrak{Tg} \frac{\omega}{2} \right], \\ M_{\text{mitte}} = \frac{P(l+a)}{4\omega} \mathfrak{Tg} \frac{\omega}{2}. \end{array} \right.$$

Man nehme folgendes Zahlenbeispiel:

317^a

X Kg.

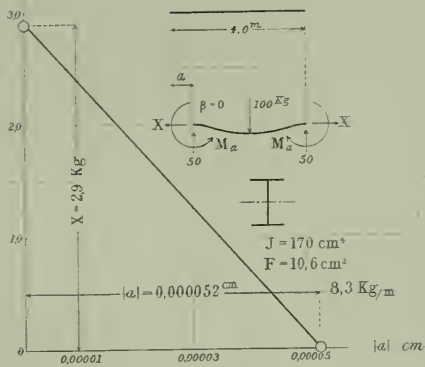


Fig. 22.

X Kg.

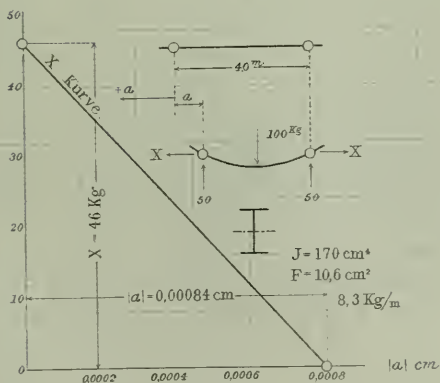


Fig. 23.

über.

α muss notwendig kleiner sein als $\frac{P^2 \gamma^5}{960 E^2 J}$ [vgl. Gl. (11)].
Ferner liefern Gl. (4), (5)

$$(29) \quad \left\{ \begin{array}{l} y_{\text{mitte}} = \frac{P(l+\alpha)}{4X} \left[1 - \frac{\gamma_g \omega}{\omega} \right], \\ M_{\text{mitte}} = \frac{P(l+\alpha)}{4\omega} \gamma_g \omega. \end{array} \right.$$

Nimmt man in dem in §4 angegebenen Doppel-T-Träger $\alpha = -0,0002$ cm an, so erhält man aus Gl. (28)

$$X = 35 \text{ kg.}$$

Die Kraft X lässt sich also, wenn α ein Viertel seines Grenzwertes $0,00084$ cm [s. § 4] beträgt, erst auf $\frac{35}{46} = \frac{1}{1,3}$ von ihrem Grösstwert vermindern.

Man bekommt ferner

$$y_{\text{mitte}} = 0,37345 \text{ cm}$$

$$M_{\text{mitte}} = 9986,93 \text{ cmkg.}$$

In Fig. 23 ist die X -Kurve als Funktion der Verschiebung α dargestellt. Man erkennt daraus, dass die Gelenkkraft mit der Verschiebung etwa gleichmässig abnimmt.

§ 13. Die Mauer ist sowohl für wagerechte Verschiebung als für Verdrehung des Trägers in gewissem Grade nachgiebig [Fig. 24].

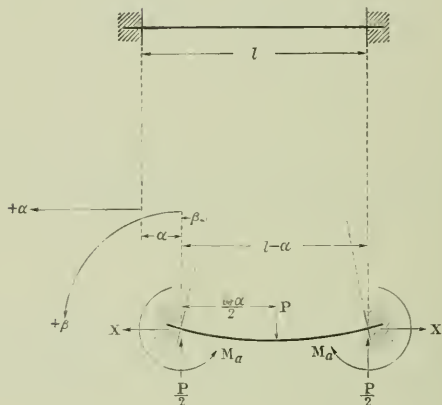


Fig. 24.

Gln. (5_a), (9) und (8_a) gelten hierbei gänzlich. Es sei beispielsweise $\alpha = -0,00002$ cm $\beta = -0,002$.

Mit den in §4 angegebenen Zahlenwert berechnet man

$$N = 27 \text{ kg}$$

$$y_{\text{mitte}} = 0,29331 \text{ cm}$$

$$M_{\text{mitte}} = 8566,93 \text{ cmkg.}$$

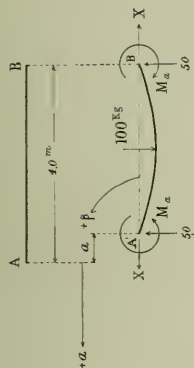
In bezug auf §7 wird

$$\frac{0,29331}{0,093368} = 3,141440 \text{ für } y_{\text{mitte}}$$

$$\frac{8566,93}{4999,86} = 1,713434 \text{ für } M_{\text{mitte}}$$

Der Träger erfährt also in der Mitte eine dreimal so grosse Senkung wie die eines entsprechenden vollkommen eingespannten Trägers, während das Moment auch etwa das Doppelte beträgt. In Vergleich mit dem Beispiel in §6 ist X hier ungefähr zehnmal so gross gerechnet, während es, da das Trägerende einer gewissen wagerechten Verschiebung unterworfen ist, im Gegenteil einen kleineren Wert erhalten müsste. Es ist leicht zu erkennen, dass dies nur durch die Voraussetzung $\beta = -0,002$ hervorgebracht ist; die Verdrehung β kann ja einen bedeutenden Einfluss auf die Vermehrung der Gelenkkraft ausüben.

I



$$J = 170 \text{ cm}^4$$

$$F = 10.6 \text{ cm}^2$$

$$8.3 \text{ kg/m}$$

$$E = 2100000 \text{ kg/cm}^2$$

$$l = 4.0 \text{ m} \quad P = 100 \text{ kg.}$$

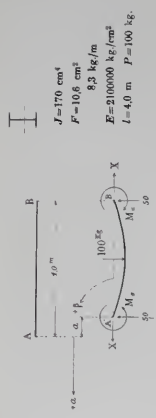
nisse
geben

Ende A in gewissem Grade verschiebbar u. drehbar	Ende A in gewissem Grade verdrehbar	Ende A wagrecht vollkommen verschiebbar	Ende A in gewissem Grade verschiebbar	Ende A vollkommen starr
$\alpha = -0.00002$ $\beta = -0.002$	$\alpha = 0$ $\beta = -0.001$	$\beta = 0$ $X = 0$	$\beta = 0$ $\alpha = -0.00003$	$\alpha = 0$ $\beta = 0$
0,29331	0,19332	0,093371	0,093369	0,093368
8566,93	6784,04	5000,0	4999,94	4999,86
27	12	0	1,2	2,9

3-8

§ 14. Zusammenfassung der Rechnungsergebnisse.

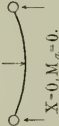
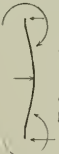
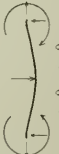
Der Übersichtlichkeit halber mögen die genommene Ergebnisse der angenommenen Zahlenbeispiele folgendermassen angegeben werden :



	Gelenk A wagrecht vollkommen verschiebbar	Gelenk A in gewissem Grade verschiebbar	Gelenk A vollkommen starr	Ende A in gewissem Grade verschiebbar u. drehbar	Ende A in gewissem Grade verdrehbar	Ende A wagrecht vollkommen verschiebbar	Ende A in gewissem Grade verschiebbar	Ende A vollkommen starr
Y_{stat} cm	0,37348	0,37345	0,37280	0,29331	0,19332	0,093371	0,093369	0,093368
M_{stat} cm kg.	10000,0	9986,93	9982,87	8566,93	6784,04	5000,0	4999,94	4999,86
X kg.	0	3,5	46	27	12	0	1,2	2,9

Um den Vergleich zu erleichtern fügen wir noch die folgende Tafel hinzu: In der Tafel z. B. in der Angabe I sind y_{mitte} M_{mitte} für den Fall $M_a=0$, $X=0$ als Einheit angenommen.

Die Zahl in der Tafel ist natürlich der vorigen Tabelle entsprechend aufzufassen.

I	Gelenk A wagerecht vollkommen verschiebbar  $X=0, M_a=0$	Y_{mitte}	1,0	0,999920	0,998179	0,785343	0,517618	0,250003	0,249997	0,249995
		M_{mitte}	1,0	0,998693	0,998287	0,856693	0,678404	0,50000	0,499994	0,499986
II	Ende A wagerecht vollkommen verschiebbar.  $X=0, \beta=0$	Y_{mitte}	3,999957	3,999636	3,992674	3,141339	2,070450	1,0	0,999979	0,999968
		M_{mitte}	2,000000	1,997386	1,996574	1,713386	1,356808	1,0	0,999988	0,999972
III	Ende A vollkommen starr.  $\alpha=0, \beta=0$	Y_{mitte}	4,000086	3,999764	3,992803	3,141440	2,070517	1,000032	1,000011	1,0
		M_{mitte}	2,000056	1,997442	1,996630	1,713434	1,356846	1,000028	1,000016	1,0

§ 15. Nachbemerkungen. In den vorhergehenden Paragraphen betrachteten wir die Verschiebungen α , β als beliebig angegebene Grössen und zwar als solche, welche sowohl von einander als auch von den äusseren Kräften, wenn darin X , M_a eingeschlossen sind, ganz unabhängig wären. Dies trifft in Wirklichkeit natürlich nicht zu, wenn der Träger mit der Mauer gewissen Verschiebungen unterliegt werden. Eine nähere Erklärung der Wirklichkeit könnte also nur dadurch erfolgen, dass man die Verschiebungen α , β als Funktionen aller übrigen Grössen betrachtet.

Am einfachsten möge

$$(30) \quad \left\{ \begin{array}{l} \alpha = k_1 X \\ \beta = k_2 M_a \end{array} \right.$$

gesetzt werden, wenn unter k_1 , k_2 zwei Erfahrungszahlen verstanden sind. Bei weiterer Umformung der Theorie in dieser Hinsicht kann die vorliegende Arbeit noch als Grundlage genommen werden.

Unsere Formeln müssen daher in sich unvollkommen sein, sie bedürfen einer weiteren Behandlung. Um die Formeln noch in Anwendung zu bringen, müssen die Verschiebungen α , β im einzelnen Fall mit Instrument oder irgendwelchen Messwerkzeugen hinreichend genau beobachtet werden.

IV. Der Träger ist gleichmassig beschwert.

§ 16. Allgemeine Gleichungen. Schliesslich mögen die Formeln für den Fall beigelegt werden, in dem sich die Einzellast P über die ganze Spannweite verteilen lässt [Fig. 25].

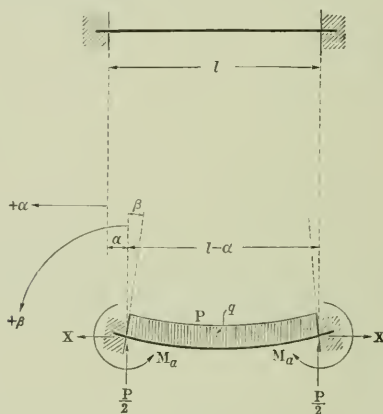


Fig. 25.

Es ist

$$q = \frac{P}{l + \alpha}.$$

Analog zu §1 erhält man für die elastische Linie die Gleichung

$$\left\{ \begin{array}{l} y = C_1 e^{\frac{2\omega x}{l+\alpha}} + C_2 e^{\frac{-2\omega x}{l+\alpha}} + \frac{Px}{2X} - \frac{Qx^2}{2X} - \frac{M_a}{X} - \frac{QKJ}{X^2}, \\ \text{worin} \\ \omega = \frac{l+\alpha}{2} \sqrt{\frac{X}{KJ}}. \end{array} \right.$$

Aus den Bedingungen $y=0$ für $x=0$ und $\frac{dy}{dx}=0$ für $x=\frac{l+\alpha}{2}$ berechnet man

$$\left\{ \begin{array}{l} C_1 = \frac{XM_a + QKJ}{2X^2 \mathfrak{C}_{\text{os } \omega}} [\mathfrak{C}_{\text{os } \omega} - \mathfrak{S}_{\text{in } \omega}], \\ C_2 = \frac{XM_a + QKJ}{2X^2 \mathfrak{C}_{\text{os } \omega}} [\mathfrak{C}_{\text{os } \omega} + \mathfrak{S}_{\text{in } \omega}]. \end{array} \right.$$

Es folgt somit:

$$(31) \quad y = \frac{Px}{2X} \left[1 - \frac{x}{l+\alpha} \right] - \left[\frac{M_a}{X} + \frac{PKJ}{X^2(l+\alpha)} \right] \left[1 - \frac{\mathfrak{C}_{\text{os}} \left(\frac{2x\omega}{l+\alpha} - \omega \right)}{\mathfrak{C}_{\text{os } \omega}} \right]$$

und

$$(32) \quad \left\{ \begin{array}{l} M = \frac{PKJ}{X(l+\alpha)} \left[1 - \frac{\mathfrak{C}_{\text{os}} \left(\frac{2x\omega}{l+\alpha} - \omega \right)}{\mathfrak{C}_{\text{os } \omega}} \right] - \frac{M_a \mathfrak{C}_{\text{os}} \left(\frac{2x\omega}{l+\alpha} - \omega \right)}{\mathfrak{C}_{\text{os } \omega}}, \\ Q = - \frac{\mathfrak{S}_{\text{in}} \left(\omega - \frac{2x\omega}{l+\alpha} \right)}{\mathfrak{C}_{\text{os } \omega}} \left[\frac{P}{2\omega} + \frac{2M_a \omega}{l+\alpha} \right]. \end{array} \right.$$

Ferner bekommt man zwischen den Verschiebungen α, β und übrigen gegebenen Grössen die zwei Gleichungen:

$$(33) \left\{ \begin{aligned} & \frac{P^2}{96 \Xi \sin^2 \omega} \left[4 \Xi \sin^2 \omega - \frac{9 \Xi \sin 2\omega}{\omega} + \frac{24 \Xi \sin^2 \omega}{\omega^2} - 6 \right] \\ & - \frac{P X \beta}{8 \Xi \sin^2 \omega} \left[2 + \frac{\Xi \sin 2\omega}{\omega} - \frac{4 \Xi \sin^2 \omega}{\omega^2} \right] \\ & + \frac{X^2 \beta^2}{8 \Xi \sin^2 \omega} \left[\frac{\Xi \sin 2\omega}{\omega} - 2 \right] - \frac{X^3}{KF} + \frac{\alpha X^2}{l + \alpha} = 0 \\ & M_a = \frac{P(l + \alpha)}{4\omega^2} \left[\frac{\omega}{\mathfrak{T} g \omega} - 1 \right] + \frac{\beta X(l + \alpha)}{2\omega \mathfrak{T} g \omega}. \end{aligned} \right.$$

§ 17. Der Träger ist vollkommen eingespannt. Man nehme als äussere Kraft nur die verteilte Last q an; X , M_a sind als von der erfolgten Formänderung herrührend angesehen. Dann gelten die obere Gleichungen, wenn darin $\alpha=0$, $\beta=0$ gesetzt ist. Sie rechnen sich um zu:

$$(34) \left\{ \begin{aligned} & \frac{P^2}{96 \Xi \sin^2 \omega} \left[4 \Xi \sin^2 \omega - \frac{9 \Xi \sin 2\omega}{\omega} + \frac{24 \Xi \sin^2 \omega}{\omega^2} - 6 \right] - \frac{X^3}{KF} = 0, \\ & M_a = \frac{Pl}{4\omega^2} \left[\frac{\omega}{\mathfrak{T} g \omega} - 1 \right], \\ & \text{wenn} \\ & \omega = \frac{l}{2} \sqrt{\frac{X}{KJ}}. \end{aligned} \right.$$

Die erstere Gleichung gestaltet sich weiter in

$$(35) \quad \frac{1}{\omega^6 \Xi \sin^2 \omega} \left[4 \Xi \sin^2 \omega - \frac{9 \Xi \sin 2\omega}{\omega} + \frac{24 \Xi \sin^2 \omega}{\omega^2} - 6 \right] - \frac{6144 K^2 J^3}{P^2 F l^6} = 0.$$

Das erste Glied ist in der Tabelle 4 zahlenmässig angegeben.

Setzt man in Gl. (31)

$$M_a = \frac{Pl}{4\omega^2} \left[\frac{\omega}{\Im g \omega} - 1 \right], \quad \alpha = 0,$$

so erhält man

$$(36) \quad y = \frac{P}{2X} \left[x \left(1 - \frac{x}{l} \right) - \frac{l \Im \sin \left(\frac{x\omega}{l} \right) \Im \sin \left(\omega - \frac{x\omega}{l} \right)}{\omega \Im \sin \omega} \right]$$

und ferner

$$(37) \quad y_{\text{mitte}} = \frac{Pl}{8X} \left[1 - \frac{\Im g \frac{\omega}{2}}{\frac{\omega}{2}} \right].$$

Der Ausdruck für M [Gl. (32)] vereinfacht sich in

$$(38) \quad M = \frac{Pl}{4\omega^2} \left[1 - \frac{\omega \Im \cos \left(\frac{2x\omega}{l} - \omega \right)}{\Im \sin \omega} \right].$$

Daraus folgt:

$$(39) \quad M_{\text{mitte}} = \frac{Pl}{4\omega^2} \left[1 - \frac{\omega}{\Im \sin \omega} \right].$$

Es sei beispielsweise $q = \frac{100}{400} = 0,25$ kg./cm.

Mit den Zahlenwerten in §4, berechnet sich

$$X = 0,74 \text{ kg,}$$

$$y_{\text{mitte}} = 0,04662 \text{ cm,}$$

$$M_{\text{mitte}} = 1666,64 \text{ cmkg.}$$

In Vergleich mit den Ergebnissen in § 7 vermindern sich die

Kraft X etwa auf 25 v. H., die Mittelsenkung etwa auf 50 v. H. und das Moment M_{mitte} auf 30 v. H.

Man nehme jetzt bei den in § 4 angegebenen Werten von J , F , E , q die Grösse l als veränderlich an. Die X -Kurve ist in Fig. 27 für $l=400$, 500, 600 und 700 cm entworfen.

Die Vergrösserung der Spannweite hat eine grosse Zunahme der Kraft X zur Folge.

§ 18. Eines der Einmauerungsenden ist wagerecht verschiebbar [Fig. 26].

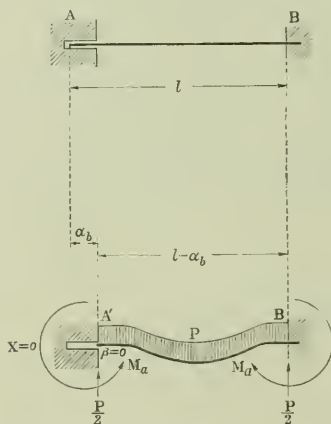


Fig. 26.

Es ist $\beta=0$, $X=0$. Mit Bezug auf Gl. (33) erhält man

324^a

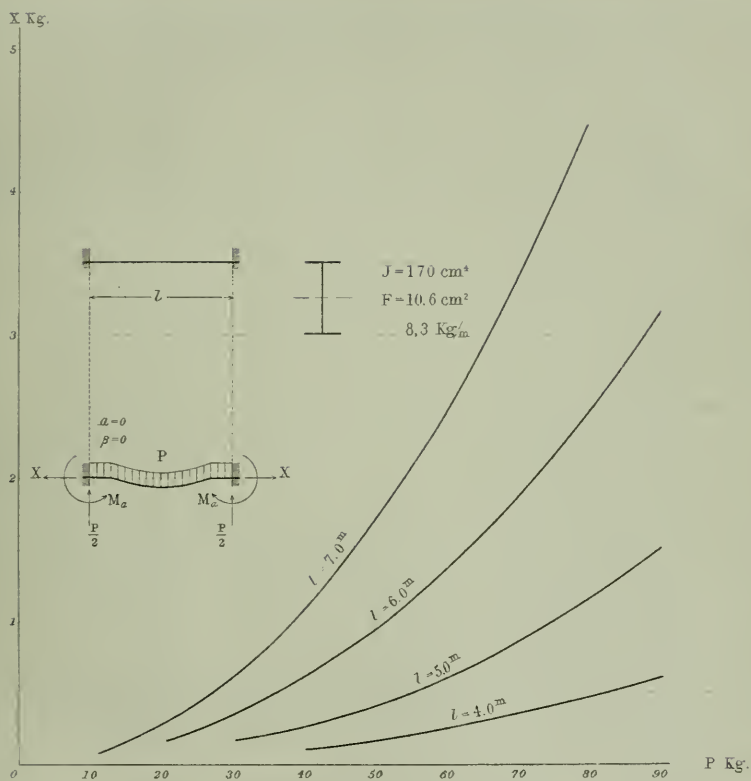


Fig. 27.

$$(40) \quad \left\{ \begin{array}{l} a_b \cong \frac{P^2 l^5}{60480 E^2 J^2}, \\ M_a \cong \frac{Pl}{12}. \end{array} \right.$$

Gln. (37), (39) liefern

$$(41) \quad \left\{ \begin{array}{l} y_{\text{mitte}} = \lim_{x \rightarrow 0} \frac{P(l-a_b)}{8X^2} \left[1 - \frac{\mathfrak{Tg} \frac{\omega}{2}}{\frac{\omega}{2}} \right] = \frac{Pl^3}{384EJ} \left[1 - \frac{P^2 l^4}{60480 E^2 J^2} \right], \\ M_{\text{mitte}} = \lim_{x \rightarrow 0} \frac{P(l-a_b)}{4\omega^2} \left[1 - \frac{\omega}{\mathfrak{S} \sin \omega} \right] = \frac{Pl}{24} \left[1 - \frac{P^2 l^4}{60480 E^2 J^2} \right]. \end{array} \right.$$

Mit den in §4 angegebenen Zahlenwerten berechnet sich

$$a_b = 0,000013 \text{ cm},$$

$$M_a = 3333,33 \text{ cmkg},$$

$$y_{\text{mitte}} = 0,04669 \text{ cm},$$

$$M_{\text{mitte}} = 1666,67 \text{ cmkg}.$$

Mit Rücksicht auf §4 sieht man, dass diese Grössen im grossen eine beträchtliche Verminderung erfahren; der Träger ist also bei gleichmässig verteilter Last weniger beansprucht, als sich die Last in der Trägermitte konzentrieren lässt.

Es möge die folgende Tafel von den Rechnungsergebnissen angegeben werden:

3 26^a



$$J=170 \text{ cm}^4$$

$$F=10,6 \text{ cm}^2$$

$$8,3 \text{ kg/m}$$

$$E=2100000 \text{ kg/cm}^2$$

$$l=4,0 \text{ m}$$

$$P=100 \text{ kg.}$$

$\alpha=0$ $\beta=0$	$X=0$ $\beta=0$	$\alpha=0$ $\beta=0$	$X=0$ $\beta=0$
0	0,000052	0	0,000013
0,093368	0,093371	0,04662	0,04669
4999,86	5000,00	1666,64	1666,67
2,9	0	0,74	0
4999,86	5000,00	3333,13	3333,33

Die zahlreichen Formeln, zu denen wir in den vorhergehenden Paragraphen gelangten, enthalten alle transzendente Grössen und gestatten leider keinen leicht erfassbaren Schluss zu ziehen. Ich begnüge mich also mit der Angabe der übersichtlichen graphischen Darstellungen, die ich in manchen Stellen für einen besonderen Träger aufgestellt hatte: [Fig. 28].

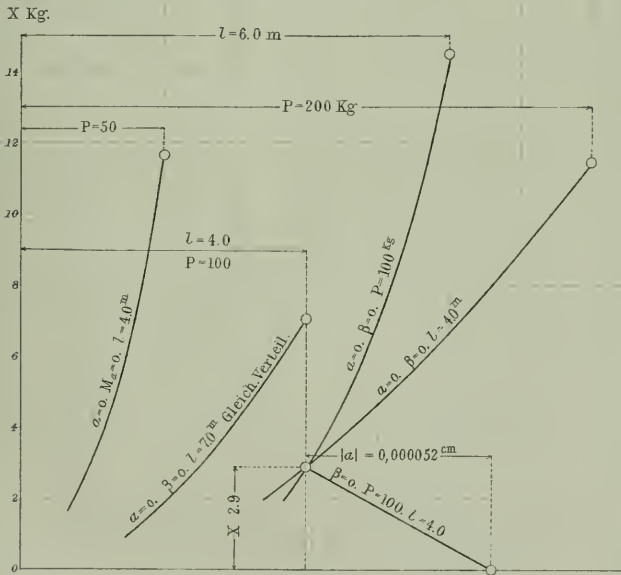


Fig. 28.

Am Schlusse dieser Arbeit ist es mir angenehme Pflicht Herrn Dr. K. Hayashi, Professor an der kaiserlichen Kyushu-Universität, für den mir bereitwilligst gewährten Rat und mannigfache Unterstützung meinen wärmsten Dank zu sagen.

Fukuoka, den 20. Februar 1922.

Fukuhei Takabeya.

Tabelle 3.

ω	$\sin \alpha - \omega$ $\omega^5 \zeta \cos^2 \frac{\alpha}{2}$	$\sin 2\alpha - 2\omega$ $\omega^3 \zeta \sin^2 \alpha$	ω	$\sin \alpha - \omega$ $\omega^5 \zeta \cos^2 \frac{\alpha}{2}$	$\sin 2\alpha - 2\omega$ $\omega^3 \zeta \sin^2 \alpha$
0	∞	∞			
0,001	166666,6333	1333333,6000	0,051	64,0446	512,4456
2	41666,6333	333333,6000	52	61,6038	492,4456
3	18518,4851	148148,4148	53	59,2698	474,4871
4	10416,6333	83333,6000	54	57,1226	457,0697
5	6666,6333	53333,1733	55	55,0631	440,5937
6	4629,5963	37036,8922	56	53,1129	424,9924
7	3401,3272	27210,7068	57	51,2645	410,1742
8	2604,6333	20337,1443	58	49,5109	396,1758
9	2037,5798	16460,7249	59	47,8437	382,8540
0,010	1666,6333	13333,1600	0,060	46,2620	370,1972
11	1377,3771	11019,1042	61	44,7575	358,1489
12	1157,3741	9259,0823	62	43,3243	346,6832
13	986,1600	7889,3695	63	41,9588	335,7592
14	850,3068	6802,5435	64	40,6568	325,3432
15	740,7074	5925,7475	65	39,4144	315,4042
16	651,0083	5208,1545	66	38,2281	305,9135
17	576,6679	4613,4330	67	37,0945	296,8447
18	514,3700	4115,0185	68	36,0105	288,1730
19	461,6472	3693,2668	69	34,9733	279,8756
0,020	416,6333	3233,1559	0,070	33,9803	271,9310
21	377,8956	3023,2540	71	33,0289	264,3201
22	341,3193	2754,6431	72	32,1169	257,0240
23	315,0265	2520,3011	73	31,2421	250,0256
24	289,3185	2314,6371	74	30,4025	243,3091
25	265,6333	2133,1556	75	29,5963	236,8594
26	246,5150	1972,2089	76	28,8217	230,6626
27	228,5903	1828,8116	77	28,0771	224,7057
28	212,5517	1700,5045	78	27,3609	218,9764
29	198,1434	1585,2364	79	26,6718	213,4643
0,030	185,1518	1481,3038	0,080	26,0084	208,1557
31	173,2971	1387,2659	81	25,3693	203,0434
32	162,7271	1301,9055	82	24,7535	198,1106
33	153,0123	1224,1871	83	24,1598	193,3677
34	144,1420	1153,2280	84	23,5873	188,7869
35	136,0211	1088,2376	85	23,0347	184,3668
36	128,5675	1028,6288	86	22,5014	180,1000
37	121,7100	973,7692	87	21,9863	175,9795
38	115,3868	923,1832	88	21,4887	171,9987
39	109,5437	876,4385	89	21,0078	168,1513
0,040	104,1333	833,1556	0,090	20,5428	164,4315
41	99,1140	793,0009	91	20,0931	160,8336
42	94,4489	755,6801	92	19,6579	157,3524
43	90,1055	720,9328	93	19,2368	153,9828
44	86,0548	688,5275	94	18,8289	150,7211
45	82,2712	658,2585	95	18,4339	147,5602
46	78,7316	629,9412	96	18,0512	144,4984
47	75,4156	603,4137	97	17,6802	141,5308
48	72,3046	578,5264	98	17,3206	138,6535
49	69,3822	555,1465	99	16,9718	135,8630
50	66,6333	533,1556	0,100	16,6334	133,1558
∞			∞	0	0

Tabelle 4.

ω	$4 \sin^2 \omega + \frac{24 \sin^4 \omega}{\omega^2} - \frac{9 \sin^2 \omega}{\omega} - 6$	ω	$4 \sin^2 \omega + \frac{24 \sin^4 \omega}{\omega^2} - \frac{9 \sin^2 \omega}{\omega} - 6$
0	∞		
0,001	25 396,8	0,051	9,76
2	6 349,2	52	9,39
3	2 821,9	53	9,04
4	1 587,3	54	8,70
5	1 015,9	55	8,39
6	705,5	56	8,09
7	518,3	57	7,81
8	386,8	58	7,54
9	313,5	59	7,29
0,010	254,0	0,060	7,05
11	209,9	61	6,82
12	176,4	62	6,60
13	150,3	63	6,39
14	129,6	64	6,20
15	112,9	65	6,01
16	99,2	66	5,83
17	87,9	67	5,65
18	78,4	68	5,49
19	70,3	69	5,33
0,020	63,5	0,070	5,18
21	57,6	71	5,03
22	52,5	72	4,89
23	48,0	73	4,76
24	44,1	74	4,63
25	40,6	75	4,51
26	37,6	76	4,39
27	34,8	77	4,28
28	32,4	78	4,12
29	30,2	79	4,06
0,030	28,2	0,080	3,96
31	26,4	81	3,87
32	24,8	82	3,77
33	23,3	83	3,68
34	22,0	84	3,59
35	20,7	85	3,51
36	19,6	86	3,43
37	18,5	87	3,35
38	17,6	88	3,27
39	16,7	89	3,20
0,040	15,9	0,090	3,12
41	15,1	91	3,06
42	14,4	92	3,00
43	13,8	93	2,93
44	13,1	94	2,87
45	12,5	95	2,81
46	12,0	96	2,75
47	11,5	97	2,69
48	11,0	98	2,64
49	10,6	99	2,59
0,050	10,2	0,100	2,53
		∞	0

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Formelübersicht.

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$$\alpha = \frac{l+z}{2} \sqrt{\frac{N}{KJ}}$$



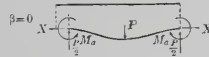
$$X\text{-Gleichung: } \frac{2 + \cos \alpha - 3 \sin \alpha}{\omega^3 \cos^3 \frac{\alpha}{2}} - \frac{8 \beta KJ (\sin \alpha - \alpha)}{P^2 (l+z)^3 \omega^3 \sin^2 \alpha} + \frac{32 \beta^2 K^2 J^2 (\sin 2\alpha - 2\alpha)}{P^2 (l+z)^3 \omega^3 \sin^2 \alpha} - \frac{1024 K^2 J^3}{l^4 (l+z)^2 P} + \frac{256 \beta K^2 J^2}{P^2 (l+z)^3 \omega^3} = 0$$

$$y = \frac{P}{2N} \left[x - \frac{(l+z) \sin \frac{2\alpha}{l+z} \cos \left(\frac{\alpha}{2} - \frac{\omega x}{l+z} \right)}{\omega \cos \frac{\alpha}{2}} + \frac{\beta (l+z) \sin \frac{2\alpha}{l+z} \sin \left(\frac{2\alpha}{l+z} - \alpha \right)}{\omega \sin \alpha} \right]$$

$$y_{mitte} = \frac{P(l+z)}{2\omega N} \left[\frac{\omega}{2} - \sin \frac{\alpha}{2} \right] - \frac{\beta (l+z) \sin \frac{\alpha}{2}}{2\omega}$$

$$M = \frac{P(l+z) \sin \left(\frac{2\alpha}{l+z} - \frac{\omega}{2} \right)}{4\omega \cos \frac{\alpha}{2}} - \frac{2\beta KJ \cos \left(\alpha - \frac{2\alpha}{l+z} \right)}{(l+z) \sin \alpha}$$

$$M_a = \frac{2\omega KJ^2}{(l+z) \sin \alpha} + \frac{P(l+z)}{4\omega} \sin \frac{\alpha}{2} \quad Q = \frac{P \cos \left(\frac{2\alpha}{l+z} - \frac{\omega}{2} \right)}{2 \cos \frac{\alpha}{2}} + \frac{\beta N \sin \left(\alpha - \frac{2\alpha}{l+z} \right)}{\sin \alpha}$$

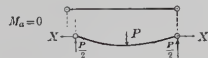


$$X\text{-Gleichung: } \frac{2 + \cos \alpha - 3 \sin \alpha}{\omega^3 \cos^3 \frac{\alpha}{2}} - \frac{1024 K^2 J^3}{P^2 (l+z)^3} + \frac{256 \beta K^2 J^2}{P^2 (l+z)^3 \omega^2} = 0$$

$$y = \frac{P}{2N} \left[x - \frac{(l+z) \sin \frac{2\alpha}{l+z} \cos \left(\frac{\alpha}{2} - \frac{\omega x}{l+z} \right)}{\omega \cos \frac{\alpha}{2}} \right]$$

$$y_{mitte} = \frac{P(l+z)}{2\omega N} \left[\frac{\omega}{2} - \sin \frac{\alpha}{2} \right] \quad M = \frac{P(l+z)}{4\omega} \sin \left(\frac{2\alpha}{l+z} - \frac{\omega}{2} \right)$$

$$M_a = \frac{P(l+z)}{4\omega} \sin \frac{\alpha}{2} \quad Q = \frac{P \cos \left(\frac{2\alpha}{l+z} - \frac{\omega}{2} \right)}{2 \cos \frac{\alpha}{2}}$$

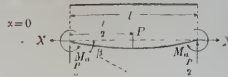


$$X\text{-Gleichung: } \frac{2 + \cos 2\alpha - 3 \sin 2\alpha}{\omega^3 \cos^3 \frac{\alpha}{2}} - \frac{1024 K^2 J^3}{P^2 (l+z)^3} + \frac{256 \beta K^2 J^2}{P^2 (l+z)^3 \omega^2} = 0$$

$$y = \frac{P}{2N} \left[x - \frac{(l+z) \sin \frac{2\alpha}{l+z} \cos \left(\frac{\alpha}{2} - \frac{\omega x}{l+z} \right)}{2\omega \cos \frac{\alpha}{2}} \right] \quad y_{mitte} = \frac{P(l+z)}{4N} \left[1 - \sin \frac{\alpha}{2} \right]$$

$$M = \frac{P(l+z)}{4\omega} \sin \left(\frac{2\alpha}{l+z} \right) \quad M_a = 0 \quad Q = \frac{P \cos \left(\frac{2\alpha}{l+z} \right)}{2 \cos \frac{\alpha}{2}}$$

$$\alpha = \frac{l}{2} \sqrt{\frac{N}{KJ}}$$



X-Gleichung:

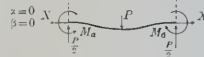
$$\frac{2 + \cos \alpha - 3 \sin \alpha}{\omega^3 \cos^3 \frac{\alpha}{2}} - \frac{8 \beta KJ (\sin \alpha - \alpha)}{P^2 \omega^3 \cos^3 \frac{\alpha}{2}} + \frac{32 \beta^2 K^2 J^2 (\sin 2\alpha - 2\alpha)}{P^2 \omega^3 \cos^3 \frac{\alpha}{2}} - \frac{1024 K^2 J^3}{P^2 \omega^3 \cos^3 \frac{\alpha}{2}} = 0$$

$$y = \frac{P}{2N} \left[x - \frac{l \sin \frac{\alpha x}{l} \cos \left(\frac{\alpha}{2} - \frac{\omega x}{l} \right)}{\omega \cos \frac{\alpha}{2}} + \frac{\beta l \sin \frac{\alpha x}{l} \sin \left(\frac{\alpha x}{l} - \alpha \right)}{\omega \sin \alpha} \right]$$

$$y_{mitte} = \frac{Pl}{2\omega N} \left[\frac{\omega}{2} - \sin \frac{\alpha}{2} \right] - \frac{\beta l \sin \frac{\alpha}{2}}{2\omega \cos \frac{\alpha}{2}}$$

$$M = \frac{Pl \sin \left(\frac{2\alpha}{l} - \frac{\omega}{2} \right)}{4\omega \cos \frac{\alpha}{2}} - \frac{2\beta \omega KJ \cos \left(\alpha - \frac{2\alpha}{l} \right)}{l \sin \alpha}$$

$$M_a = \frac{2\omega KJ^2}{l \sin \alpha} + \frac{Pl}{4\omega} \sin \frac{\alpha}{2} \quad Q = \frac{P \cos \left(\frac{2\alpha}{l} - \frac{\omega}{2} \right)}{2 \cos \frac{\alpha}{2}} + \frac{\beta N \sin \left(\alpha - \frac{2\alpha}{l} \right)}{\sin \alpha}$$

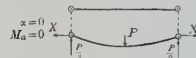


$$X\text{-Gleichung: } \frac{2 + \cos \alpha - 3 \sin \alpha}{\omega^3 \cos^3 \frac{\alpha}{2}} - \frac{1024 K^2 J^3}{P^2 \omega^3} = 0$$

$$y = \frac{P}{2N} \left[x - \frac{l \sin \frac{\alpha x}{l} \cos \left(\frac{\alpha}{2} - \frac{\omega x}{l} \right)}{\omega \cos \frac{\alpha}{2}} \right]$$

$$y_{mitte} = \frac{Pl}{2\omega N} \left[\frac{\omega}{2} - \sin \frac{\alpha}{2} \right] \quad M = \frac{Pl}{4\omega} \sin \left(\frac{2\alpha}{l} - \frac{\omega}{2} \right)$$

$$M_a = \frac{Pl}{4\omega} \sin \frac{\alpha}{2} \quad Q = \frac{P \cos \left(\frac{2\alpha}{l} - \frac{\omega}{2} \right)}{2 \cos \frac{\alpha}{2}}$$



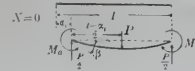
X-Gleichung:

$$\frac{2 + \cos 2\alpha - 3 \sin 2\alpha}{\omega^3 \cos^3 \frac{\alpha}{2}} - \frac{1024 K^2 J^3}{P^2 \omega^3} = 0$$

$$y = \frac{P}{2N} \left[x - \frac{l \sin \frac{2\alpha x}{l}}{2\omega \cos \frac{\alpha}{2}} \right] \quad y_{mitte} = \frac{Pl}{4N} \left[1 - \sin \frac{\alpha}{2} \right]$$

$$M = \frac{Pl}{4\omega} \sin \left(\frac{2\alpha}{l} \right) \quad M_a = 0 \quad Q = \frac{P \cos \left(\frac{2\alpha}{l} \right)}{2 \cos \frac{\alpha}{2}}$$

$$\alpha = 0$$



X-Gleichung: $N = 0$

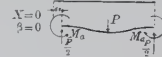
$$y = \frac{P}{16EJ} \left[z^2 (l - z_1) - \frac{4z^3}{3} \right] + \frac{\beta (z - l + z_1) z}{l - z_1}$$

$$y_{mitte} = \frac{P(l - z_1)^2}{192EJ} - \frac{\beta (l - z_1)}{4}$$

$$M = \frac{P}{2} z - \frac{P(l - z_1)}{8} - \frac{2\beta EJ}{l - z_1}$$

$$M_a = \frac{P(l - z_1)}{8} + \frac{2\beta EJ}{l - z_1}$$

$$Q = \frac{P}{2}$$



X-Gleichung: $N = 0$

$$y = \frac{P}{16EJ} \left[z^2 \left(1 - \frac{P^2 l}{15360 E^2 J^2} \right) - \frac{4z^3}{3} \right]$$

$$y_{mitte} = \frac{P^3}{192EJ} \left[1 - \frac{P^2 l}{15360 E^2 J^2} \right]$$

$$M = \frac{P}{2} z - \frac{P}{8} \left[1 - \frac{P^2 l}{15360 E^2 J^2} \right]$$

$$M_a = \frac{P}{8} \left[1 - \frac{P^2 l}{15360 E^2 J^2} \right]$$

$$Q = \frac{P}{2} \quad a_n = \frac{P^2 l}{15360 E^2 J^2}$$



X-Gleichung: $N = 0$

$$y = \frac{P}{16EJ} \left[z^2 \left(1 - \frac{P^2 l}{960 E^2 J^2} \right) - \frac{4z^3}{3} \right]$$

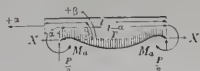
$$y_{mitte} = \frac{P^3}{48EJ} \left[1 - \frac{P^2 l}{960 E^2 J^2} \right]$$

$$M = \frac{P}{2} z \quad M_a = 0$$

$$Q = \frac{P}{2} \quad a_n = \frac{P^2 l}{960 E^2 J^2}$$

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$$\omega = \frac{l+z}{2} \sqrt{\frac{N}{Kl}}$$



X-Gleichung:

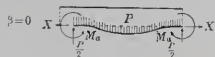
$$\frac{P^2}{96 \sin^2 \omega} \left[4 \sin^2 \omega + \frac{24 \sin^2 \omega}{\omega^2} - \frac{9 \sin 2\omega}{\omega} - 6 \right] - \frac{P^2 X^3}{8 \sin^2 \omega} \left[2 + \frac{\sin 2\omega}{\omega} - \frac{4 \sin^2 \omega}{\omega^2} \right] - \frac{X^3}{8 \sin^2 \omega} \left[2 - \frac{\sin 2\omega}{\omega} \right] - \frac{X^4}{KE} + \frac{aX^2}{l+z} = 0.$$

$$y = \frac{Pl}{2X} \left[1 - \frac{x}{l+z} \right] - \frac{(l+z) \left(\frac{P}{2X} + \beta \right) \sin \left(\frac{x\omega}{l+z} \right) \sin \left(\omega - \frac{x\omega}{l+z} \right)}{\omega \sin \omega}.$$

$$y_{mitte} = \frac{P(l+z)}{8X} \left[1 - \frac{\frac{3}{8} \frac{\omega}{2}}{\frac{\omega}{2}} \right] - \frac{9(l+z) \frac{3}{8} \frac{\omega}{2}}{2\omega}.$$

$$M = \frac{P(l+z)}{4\omega^2} \left[1 - \frac{\omega \cos \left(\frac{2x\omega}{l+z} - \omega \right)}{\sin \omega} \right] - \frac{\beta X(l+z) \cos \left(\frac{2x\omega}{l+z} - \omega \right)}{2\omega}.$$

$$M_a = \frac{P(l+z)}{4\omega^2} \left[\frac{\omega}{\sin \omega} - 1 \right] + \frac{9(l+z)X}{2\omega \sin \omega}, \quad Q = \left[\frac{P}{2} + \beta X \right] \frac{\sin \left(\omega - \frac{2x\omega}{l+z} \right)}{\sin \omega}.$$



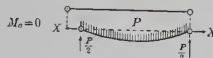
X-Gleichung:

$$\frac{P^2}{96 \sin^2 \omega} \left[4 \sin^2 \omega + \frac{24 \sin^2 \omega}{\omega^2} - \frac{9 \sin 2\omega}{\omega} - 6 \right] - \frac{X^3}{KE} + \frac{aX^2}{l+z} = 0.$$

$$y = \frac{P}{2X} \left[x \left(1 - \frac{x}{l+z} \right) - \frac{(l+z) \sin \left(\frac{x\omega}{l+z} \right) \sin \left(\omega - \frac{x\omega}{l+z} \right)}{\omega \sin \omega} \right].$$

$$y_{mitte} = \frac{P(l+z)}{8X} \left[1 - \frac{\frac{3}{8} \frac{\omega}{2}}{\frac{\omega}{2}} \right], \quad M = \frac{P(l+z)}{4\omega^2} \left[1 - \frac{\omega \cos \left(\frac{2x\omega}{l+z} - \omega \right)}{\sin \omega} \right].$$

$$M_a = \frac{P(l+z)}{4\omega^2} \left[\frac{\omega}{\sin \omega} - 1 \right], \quad Q = \frac{P}{2} \frac{\sin \left(\omega - \frac{2x\omega}{l+z} \right)}{\sin \omega}.$$



$$X\text{-Gleichung: } \frac{P^2}{96\omega^2 \cos^2 \omega} \left[6 + 24 \cos^2 \omega - 4\omega^2 \cos^2 \omega - 15 \frac{\sin 2\omega}{\omega} \right] + \frac{X^3}{KE} + \frac{aX^2}{l+z} = 0.$$

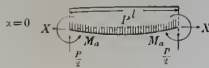
$$y = \frac{P}{2X} \left[x \left(\frac{x^2}{l+z} - \frac{l+z}{2\omega^2} \right) + \frac{P(l+z)}{4\omega^2 X} \frac{\cos \left(\frac{2x\omega}{l+z} - \omega \right)}{\cos \omega} \right].$$

$$y_{mitte} = \frac{P(l+z)}{4X} \left[\frac{1}{2} - \frac{1}{\omega^2} \left(1 - \frac{1}{\cos \omega} \right) \right], \quad M = \frac{P(l+z)}{4\omega^2} \left[1 - \frac{\cos \left(\frac{2x\omega}{l+z} - \omega \right)}{\cos \omega} \right].$$

$$M_a = 0.$$

$$Q = \frac{P}{2\omega} \frac{\sin \left(\omega - \frac{2x\omega}{l+z} \right)}{\cos \omega}.$$

$$\omega = \frac{1}{2} \sqrt{\frac{X}{Kl}}$$



X-Gleichung:

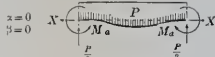
$$\frac{P^2}{96 \sin^2 \omega} \left[4 \sin^2 \omega + \frac{24 \sin^2 \omega}{\omega^2} - \frac{9 \sin 2\omega}{\omega} - 6 \right] - \frac{P^2 X^3}{8 \sin^2 \omega} \left[2 + \frac{\sin 2\omega}{\omega} - \frac{4 \sin^2 \omega}{\omega^2} \right] - \frac{X^3}{8 \sin^2 \omega} \left[2 - \frac{\sin 2\omega}{\omega} \right] - \frac{X^4}{KE} = 0.$$

$$y = \frac{Pl}{2X} \left[1 - \frac{x}{l} \right] - \frac{(l+z) \sin \left(\frac{x\omega}{l} \right) \sin \left(\omega - \frac{x\omega}{l} \right)}{\omega \sin \omega}.$$

$$y_{mitte} = \frac{Pl}{8X} \left[1 - \frac{\frac{3}{8} \frac{\omega}{2}}{\frac{\omega}{2}} \right] - \frac{9l \frac{3}{8} \frac{\omega}{2}}{2\omega}.$$

$$M = \frac{Pl}{4\omega^2} \left[1 - \frac{\omega \cos \left(\frac{2x\omega}{l} - \omega \right)}{\sin \omega} \right] - \frac{l\beta X \cos \left(\frac{2x\omega}{l} - \omega \right)}{2\omega}.$$

$$M_a = \frac{Pl}{4\omega^2} \left[\frac{\omega}{\sin \omega} - 1 \right] + \frac{9lX}{2\omega \sin \omega}, \quad Q = \left[\frac{P}{2} + \beta X \right] \frac{\sin \left(\omega - \frac{2x\omega}{l} \right)}{\sin \omega}.$$



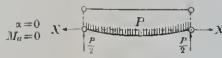
X-Gleichung:

$$\frac{P^2}{96 \sin^2 \omega} \left[4 \sin^2 \omega + \frac{24 \sin^2 \omega}{\omega^2} - \frac{9 \sin 2\omega}{\omega} - 6 \right] - \frac{X^3}{KE} = 0.$$

$$y = \frac{P}{2X} \left[x \left(1 - \frac{x}{l} \right) - \frac{l \sin \frac{x\omega}{l} \sin \left(\omega - \frac{x\omega}{l} \right)}{\omega \sin \omega} \right].$$

$$y_{mitte} = \frac{Pl}{8X} \left[1 - \frac{\frac{3}{8} \frac{\omega}{2}}{\frac{\omega}{2}} \right], \quad M = \frac{Pl}{4\omega^2} \left[1 - \frac{\omega \cos \left(\frac{2x\omega}{l} - \omega \right)}{\sin \omega} \right].$$

$$M_a = \frac{Pl}{4\omega^2} \left[\frac{\omega}{\sin \omega} - 1 \right], \quad Q = \frac{P}{2} \frac{\sin \left(\omega - \frac{2x\omega}{l} \right)}{\sin \omega}.$$



X-Gleichung:

$$\frac{P^2}{96\omega^2 \cos^2 \omega} \left[6 + 24 \cos^2 \omega - 4\omega^2 \cos^2 \omega - 15 \frac{\sin 2\omega}{\omega} \right] + \frac{X^3}{KE} = 0.$$

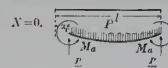
$$y = \frac{P}{2X} \left[x \left(\frac{x^2}{l} - \frac{l}{2\omega^2} \right) + \frac{Pl}{4\omega^2 X} \frac{\cos \left(\frac{2x\omega}{l} - \omega \right)}{\cos \omega} \right].$$

$$y_{mitte} = \frac{Pl}{4X} \left[\frac{1}{2} - \frac{1}{\omega^2} \left(1 - \frac{1}{\cos \omega} \right) \right], \quad M = \frac{Pl}{4\omega^2} \left[1 - \frac{\cos \left(\frac{2x\omega}{l} - \omega \right)}{\cos \omega} \right].$$

$$M_a = 0.$$

$$Q = \frac{P}{2\omega} \frac{\sin \left(\omega - \frac{2x\omega}{l} \right)}{\cos \omega}.$$

$$\omega = 0$$



X-Gleichung:

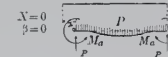
$$X = 0.$$

$$y = \frac{Pl}{24El} \left[\frac{x^4}{(l-z_1)} - 2x^3 + x^2(l-z_1) \right] - \frac{X^3}{KE} = 0.$$

$$y_{mitte} = \frac{Pl(-z_1)^3}{384El} - \frac{9(l-z_1)}{4}.$$

$$M = \frac{Pl}{2} - \frac{Pl^2}{2(l-z_1)} - \frac{l(-z_1)}{12} - \frac{2\beta El}{l-z_1}.$$

$$M_a = \frac{Pl(-z_1)}{12} + \frac{2\beta El}{l-z_1}, \quad Q = \frac{Pl}{2} - \frac{Pl^2}{l-z_1}.$$



X-Gleichung:

$$X = 0.$$

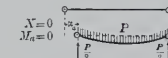
$$y = \frac{Pl}{24El} \left[\frac{x^4}{(l-z_0)} - 2x^3 + x^2(l-z_0) \right].$$

$$y_{mitte} = \frac{Pl(-z_0)^3}{384El}.$$

$$M = \frac{Pl}{2} - \frac{Pl^2}{2(l-z_0)} - \frac{l(-z_0)}{12}.$$

$$M_a = \frac{Pl(-z_0)}{12}, \quad Q = \frac{Pl}{2} - \frac{Pl^2}{l-z_0}.$$

$$a_0 = \frac{Pl^2}{60480El^2}.$$



X-Gleichung:

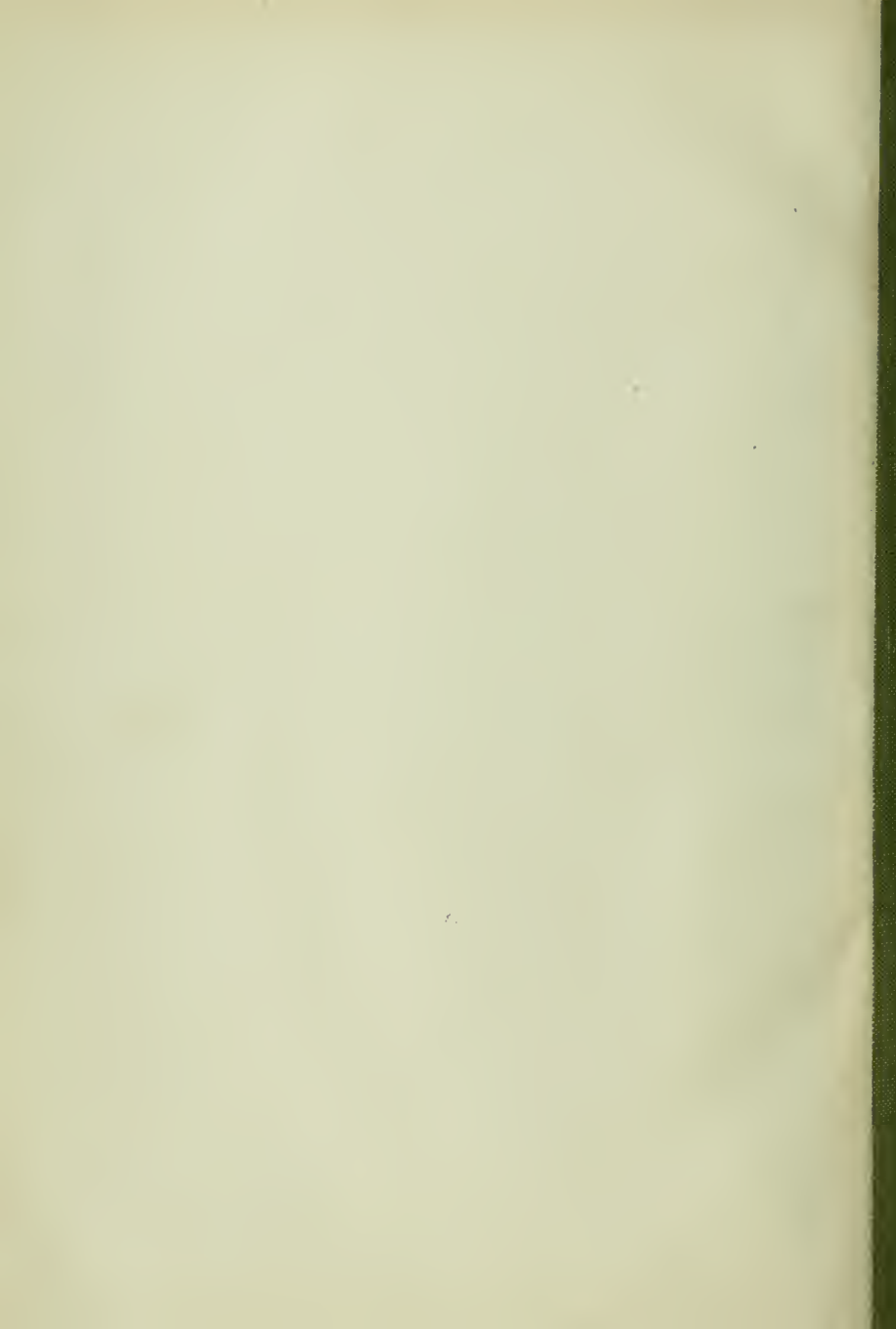
$$y = \frac{Pl}{24El} \left[\frac{x^4}{(l-z_0)} - 2x^3 + x^2(l-z_0) \right].$$

$$y_{mitte} = \frac{5Pl(-z_0)^3}{384El}.$$

$$M = \frac{Pl}{2} - \frac{Pl^2}{2(l-z_0)}.$$

$$M_a = 0, \quad Q = \frac{Pl}{2} - \frac{Pl^2}{l-z_0}.$$

$$a_0 = \frac{17Pl^2}{40320El^2}.$$



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